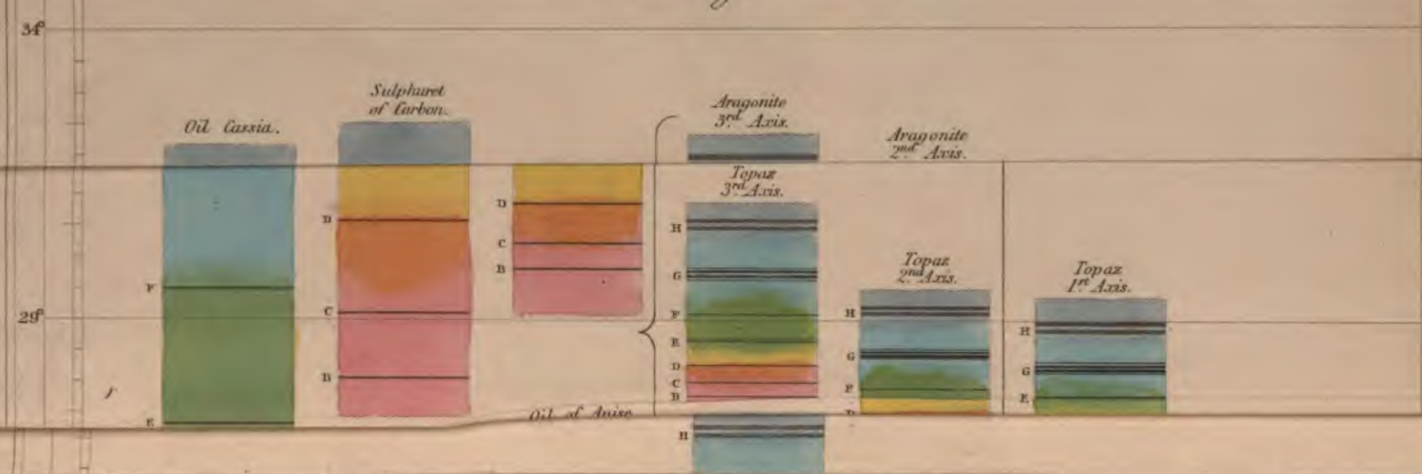
A Chart of Prismatic Spectra for various media
Prism-angle = 41°. 40′.



B. Powell, F.R.S. Del. 1841.

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A GENERAL AND ELEMENTARY VIEW

OF THE

UNDULATORY THEORY,

AS APPLIED TO THE

DISPERSION OF LIGHT,

AND

SOME OTHER SUBJECTS.

INCLUDING THE SUBSTANCE OF SEVERAL PAPERS, PRINTED IN THE PHILOSOPHICAL TRANSACTIONS, AND OTHER JOURNALS.

BY THE REV.

BADEN POWELL, M.A., F.R.S., F.G.S., F.R.A.S.

SAVILIAN PROFESSOR OF GEOMETRY IN THE UNIVERSITY OF OXFORD.

Omnis enim Philosophiæ difficultas in eo versari videtur, ut, a phenomenis MOTUUM, investigemus vires naturæ, deinde ab his viribus, demonstremus phenomena reliqua.—NEWTON, *Princip. prefat.*

LONDON:
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M.DCCC.XLI.

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TO

THE MOST NOBLE

THE MARQUIS OF NORTHAMPTON,

F.G.S., F.A.S., &c.

PRESIDENT OF THE ROYAL SOCIETY.

My Lord,

In acknowledging your Lordship's kindness in allowing me to testify my respect by inscribing this volume to you, I am anxious to refer to the circumstances under which I made the request.

My former researches on the Dispersion of Light have successively appeared in the Philosophical Transactions; and it would certainly have been most desirable that the series should be continued through the same channel. But the nature of the discussion into which I have been led, with the view of bringing the sub-

ject into a more complete shape, has been such as to render a separate publication more advisable.

By your Lordship's kindness, however, I may still consider myself as presenting these researches to the ROYAL SOCIETY, under the gratifying form of a dedication to its distinguished HEAD.

I have the honour to remain,

Your Lordship's

Most obedient and obliged Servant,

THE AUTHOR.

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ERRATA.

page	*line*	*for*	*read*
xxix	16	that	that the
xxxii	26	p. 23	Opticks, p. 23
xlviii	18	reprinted	repeated
25	6	on	or
35	18	h	k
43	8	*Mathematics*	*Mathématiques*
44	11	these	there
45	16	totile dirignée dum	totale diriguée dans
56	8	$\sin^2\theta\frac{d\xi}{dx}$	$\sin 2\theta\frac{d\xi}{dx}$
56	13	$= Q\zeta$	$+ Q\zeta$
86	15	undulating	undulatory
109	1	chromate of lead	solution of chromate of lead
...	...	potass	solution of potass
130	7	$x = b$	$x = b$ nearly
131	15	$2_B{}^2$	$l_B{}^2$

34 Art. 124, *add the following note:—*

On this point see also the note appended to Sir J. Lubbock's paper; p. 353, *Journal of Science*, Vol. xv. Nov. 1839.

INTRODUCTION.

i. THE large and rapid accessions which almost every branch of science is daily receiving, and the incessant accumulation of new researches and discoveries, entail the constant necessity for a vigilant review and connected recapitulation of them from time to time, without which it would be totally impossible in any degree to keep pace with their advance: and which is hardly less essential to the establishment of systematic science, than the discoveries themselves are to its progress.

In no instance does this remark more forcibly apply than in regard to the present state of *Physical Optics.* Such a recapitulation embracing the extent of this science would be a work of the highest value and importance, and of no small magnitude,—were it merely supplementary to what is already before the world of the same kind of a date only a few years back. An undertaking of so comprehensive a nature, however, is beyond my present design.

In *one department* of this science the want of something of this sort has peculiarly impressed itself on my attention; and in the hope of doing

what I can to supply it, I have in the following Tract attempted to take such a general review of the subject of the *dispersion of light;* embracing incidentally some few collateral topics, but essentially involving an exhibition of the fundamental principles in a form of considerable generality, and this partly as a *resumé* of some previous researches which have from time to time appeared, and partly as supplying what was wanting to complete them, at least to such an extent, as the means, whether theoretical or experimental at present in our hands, seem to allow.

ii. The investigations here pursued are based upon the principles of the undulatory theory; which has excited so much discussion within the last few years,—whether as to the stability of the whole system or as to the structure of its subordinate parts. It may therefore be desirable to say a few words on the light in which that theory in general is here regarded.

The object of the present Tract has no reference to controversial arguments either for or against the theory of undulations;—or to comparisons between it and any rival theory; or appeals to the authority of great names on either side. My views will be found to have little in common with those of writers on the one hand whose zeal leads them to uphold and idolize as perfect the physical hypothesis of a

material æther;—or of those, on the other, who (not always in the most philosophical or becoming tone) pronounce every objection insuperable, and every result really or apparently opposed to the theory as fatal to its reception.

At the same time it may be proper to explain to what extent, and on what sort of grounds, I adopt the system of undulations; and these are, I think, of a kind which must be allowed the most safe and reasonable; such, namely, as consist with an unreserved acknowledgement of the difficulties with which its reception is surrounded. The theory, in itself, is in many parts obscure, defective, and unsettled; and in its application to the phenomena is in many instances unsatisfactory, while for some classes of facts it has afforded no explanation at all. Thus on the one hand, so far from any *unqualified* support of this theory, within the compass of these remarks it will be seen that I shall have occasion to point to discrepancies between theory and fact, as well as to theoretical deficiencies which I believe *yet remain to be cleared up:* on the other hand, however, that they *will eventually* be so cleared up,—such is my conviction of the soundness of the great principles on which the theory rests,—I can entertain no kind of doubt.

If any rival theory had even suggested the possibility of such explanations as this does not furnish,

we might enter into a comparison of them. If there were any reason for believing the obstacles really insuperable on the wave theory, there might be some ground for looking to others. But if by every consideration of analogy and probability, some slight improvement in our elementary conceptions of vibratory motion be all that is wanting to overcome those difficulties, no sound philosopher would for a moment think of abandoning so hopeful a track, and none but the most ignorant or perverse would find in the obstacles which beset it anything but the more powerful stimulus to pursue it.

iii. The existence of an *alternating motion of some kind,* at minute intervals along a ray, is as real as the motion of translation by which light is propagated through space. Both must essentially be combined in any correct conception we form of light. That this alternating motion must have reference to certain directions *transverse* to that of the ray is equally established as a consequence of phenomena—and these two principles must form the basis of any explanation which can be attempted.

This, it will be seen, is distinct from insisting on the actual existence of a material fluid medium, of which we certainly have no independent evidence. We may be content to look at the theory simply as a mathematical system which faithfully represents, at least, a wide range of phenomena: and to

some extent connects the laws so made out with dynamical principles regulating the motions of a system of points, combined to form an elastic system: which for *brevity* and *illustration we call molecules*, constituting an *ætherial medium.*

The theory has evinced its powers in having often anticipated the discovery of facts, in constantly illustrating those facts which it explains, by new and enlarged views of their relations, and in a singular capacity to extend itself, and evolve new resources, as new demands are made upon it. It thus rests its claims on its sufficiency to account for large and constantly increasing classes of phenomena: and if in many instances it fail, the only course is, where it seems defective, to recur to and endeavour to remodel its first principles, and try whether by some new element, or more extended development, they may not be made to embrace the new results without impairing their application to the old.

This has been the method pursued by those eminent mathematicians who have laboured in this field with such distinguished success in the successive improvements on the theory; and it is to one of the most striking instances of this kind that the following remarks will principally refer:—the establishment of at least a *general* explanation of the dispersion by M. Cauchy. I propose to offer some illustrations of this theory both in its abstract

principles,—in its relations to certain other branches of the subject,—and more especially in its application to the numerical results of observation, with reference to the degree in which it may be actually verified by such comparison.

iv. In this place I will only remark in general, that the progress as yet made in this portion of the enquiry must be acknowledged far from being so considerable as might be wished: yet as far as it has extended it must at least be allowed to be of the highest promise.

The problem of the *dispersion* was, for a long time, confessedly the opprobrium of all theories of light, but more especially,—in proportion to its higher pretensions,—of the undulatory; not simply because it *had not explained* those phenomena, but because, according to the received views, it was *at variance* with them. It has only been within the last few years, that, by a modification of the principles of this theory, it has been brought to bear on the question of the dispersion; and vindicated, as to its principles at least, from this reproach. Yet there are still those who are so unreasonable as to inveigh against the theory, because it has not, in that comparatively short period, succeeded in clearing the subject of *all* its difficulties; and who, instead of any degree of satisfaction at what *has* been done, express only displeasure at what *has not* been

effected. Passages may be found in the writings of some philosophers of the present day, in which complaints of this nature occur, involving remarks and reflections which we cannot well set down to ignorance, and which, in any other point of view, can neither be regarded as peculiarly creditable to the taste of the writers, nor to their appreciation of what is justly due to a series of researches of such a character as must fairly be allowed to belong to those in question, even if they could be shown to have failed in their object.

Nothing, I think, can be more at variance both with philosophy and fact, than some observations of this kind in which even distinguished writers have occasionally indulged. Yet so far from being blindly devoted to the theory, I shall have occasion in the following pages to comment on points in which it seems at present materially defective, and to which the attention of those who are in a condition to grapple with the difficulties of this intricate portion of dynamical research, is at present most powerfully called.

v. But before proceeding it may not be out of place to make one or two remarks on points connected with the *history* of the theory of light which may tend to illustrate the main subject.

It has been usual with philosophical writers, up to a very recent period, to speak of the two rival

theories of Emission and Undulation,—of Newton and Huyghens; and to discuss the claims of the corpuscular or Newtonian doctrine as if it were a connected mathematical theory, developed as such and maintained by Newton in opposition to that proposed by the continental philosophers. And when the progress of discovery had compelled some of the partizans of emission to admit the superior claims of undulations, they have yet found in the authority of Newton's name a plea for their predilection, and we have heard eloquent confessions of a reluctance to abandon ground which he had consecrated, and a desire still to linger in "the temple in which Newton worshipped."

But it is remarkable how unsubstantial all this is found to be when we look into the *facts* of the case,—when we ask the question—*What was* this corpuscular theory? and in what form or to what extent *did Newton really uphold it?* I believe that no better answer to these queries can be found than the following statements will supply.

vi. In the first place, in the *Principia*, Book I. §. 14. Newton investigates what would be the paths of corpuscules in motion attracted by media through which they pass: which he then applies to the refraction and reflexion of light.

This has been justly regarded with that admiration which belongs to the first solution of a problem

of molecular forces: and undoubtedly gives a perfect explanation of the ordinary phenomena of reflexion and refraction, but extends no further.

Again, totally distinct from this last, was the conception of *fits* of easy transmission and reflexion, by which Newton represented the complex phenomena of colours of thin plates. To describe another class of facts, he came to the singular inference that a ray of light has "sides" and distinct properties referring to those sides;—but this had no relation to the foregoing. To explain the coloured fringes at the edges of shadows another sort of property was devised, and a certain bending of the rays in passing near the edges of bodies was to be imagined, "backwards and forwards with a motion like that of an eel." [Opticks; Query (3)].

In these, and perhaps some other instances, we trace Newton's corpuscular views of light such as they were: we find a corpuscular theory of reflexion and refraction: another corpuscular theory of fits, another of polarity, another of inflexion;—and the like: but these did not constitute *one* connected theory, but referred only to those particular phenomena: they were wholly unconnected by a common principle; they formed *no system.*

vii. But as to the degree of weight he attached to the corpuscular doctrine, we have the following explicit statement:—

"'Tis true that from my theory I argue the corporeity of light; but I do it without any absolute positiveness, as the word "perhaps" intimates; and make it at most but a very plausible consequence of the doctrine, and not a fundamental supposition, nor so much as any part of it." (*Phil. Trans.* Vol. x. 1675, p. 5086.)

But besides this, it is equally well known that both in the paper just quoted and in several other parts of his writings, Newton expressly supported an hypothesis of *undulations:* describing nearly in the terms used by the modern writers the nature of the supposed ætherial medium as "much of the same constitution as air, but far rarer, subtler, and more strongly elastic."...(See *Birch's History*, R.S. III. 249.) And in the queries appended to the "*Opticks*," he refers more particularly to such an elastic medium as that through which vibrations are propagated, giving rise both to light and heat. (Query 18. 22.) And yet more precisely he distinguishes the different lengths of "vibrations," manifestly using that term in the sense now restricted to "undulations:" in continuation of the passages first cited he goes on to observe, "It is to be supposed that the ether is a vibrating medium, like air, only the vibrations far more swift and minute; those of air made by a man's ordinary voice, succeeding one another at more than half a foot, or a foot distance; but those of ether at a less distance than the hun-

dred thousandth part of an inch. And as in air the vibrations are some larger than others, but yet all equally swift, (for in a ring of bells the sound of every tone is heard at two or three miles distance, in the same order that the bells are struck,) so I suppose the etherial vibrations differ in bigness, but not in swiftness." (*Birch*. III. 251.)

viii. These waves however were certainly supposed by Newton to be something different from those of Huyghens;—for he observes (in the 17th Query): "When a ray of light falls on the surface of any pellucid body, and is there refracted or reflected, may not waves of vibrations or tremors be thereby excited in the refracting or reflecting medium? and are not these vibrations propagated from the point of incidence to great distances? and do they not *overtake the rays of light*, and by overtaking them successively, do they not put them into the fits of easy reflexion and easy transmission described above?"

From this and other passages it would seem that he was desirous to combine the undulatory view with some notion of emitted rays. He also refers expressly to the ether as existing in different *density* within and without the medium, and thus causing the reflexion or refraction—or rather in some measure giving rise to the attractions and repulsions, to which, as before observed, he elsewhere had recourse

for his explanation. (See *Birch.* III. 247. *Opt. Query* 19.)

ix. Nevertheless passages may also be cited where Newton seems to oppose and urge objections against the wave theory. The main difficulty seems to have been that which, though refuted by Huyghens, has often been recurred to by opponents, viz. the *rectilinear* propagation of light passing through an aperture. "To me," Newton observes, "the fundamental supposition itself seems impossible; namely, that the waves or vibrations of any fluid can, like the rays of light, be propagated in straight lines without a continual and very extravagant spreading and bending every way into the quiescent medium where they are terminated by it. I mistake, if there be not both experiment and demonstration to the contrary." (*Phil. Trans.* Vol. x. No. 88. p. 5089.)

It is remarkable that he should have himself referred to the very circumstance which affords the solution of the difficulty,—the great difference in the lengths of waves of light and of sound.

In the Second Book of the *Principia* (§ 8. Prop. 42) he expressly demonstrates a theorem to this effect; which he observes is experimentally confirmed in the case of sound, and will not apply to light. (*Scholium*, Prop. 50.)

Again, in the 28th query of the Opticks, he asks, "Are not all hypotheses erroneous, in which light is supposed to consist in pression or motion propagated through a fluid medium? If it consisted in pression or motion propagated either in an instant or in time, it would bend into the shadow." And he proceeds to illustrate the case by the analogy of waves in water and in that of sound. "But light," he continues, "is never known to follow crooked passages or to bend into the shadow;" he then notices the small apparent bending of rays in passing the edges of bodies, (according to his view of the inflexion) which he shews is not a real exception to the remark.

x. These quotations may suffice to shew how little evidence can be found for those who desire to secure Newton as a partizan, whether to the undulationists or corpuscularians. And in the attempt to reconcile these apparently conflicting statements, various have been the suppositions:—

Some have contended that in the course and progress of his speculations and enquiries Newton several times changed his views on the subject. Others, especially M. Biot in his Life of Newton, (see translation, Library of Useful Knowledge,) have maintained that his opinions never varied, but that he upheld the wave-theory throughout; though at different times he had recourse to other illustrations, as circumstances might require.

Thus he is supposed to have originally adopted the undulatory view of refraction, as laid down in his paper of 1675, though he afterwards gave, in the Principia, in 1687, the demonstration on the principle of corpuscular attractions, as an application of that doctrine which he had then first succeeded in establishing on a mathematical basis: while on the wave-hypothesis he was not in possession of any principles on which to institute a *mathematical* investigation.

It is thus argued that his opinions throughout continued firmly established in favour of undulations, though he occasionally deviated so far as to illustrate certain parts of the subject by auxiliary hypotheses dependent on a supposed emission of material particles.

xi. Such an explanation however appears to me as unsatisfactory as the notion of his supposed devotion to the material doctrine. The truth, I believe, was, that he cannot be said to have adopted *any* theory of light. His views extended no further than to represent his inductive conclusions, and to express the laws of particular classes of phenomena.

The physical conception of the nature of light he justly regarded as a point wholly distinct from these; and the spirit in which he ever ventured to speculate upon it seems clearly expressed in his own words:—

"But whatever may be the advantages or disadvantages of this hypothesis (Hooke's), I hope I may

be excused from taking it up, since I do not think it needful to explicate my doctrine by any hypothesis at all."—(*Phil. Trans.* x. p. 5089, or *Birch*, III. 248.)

Again: "Because I have observed the heads of some great virtuosos to run much upon hypotheses, I will give one which I should be inclined to consider as the most probable, *if I were obliged to adopt one.*"................"*Were I to assume an hypothesis* it should be this if propounded more generally, so as not to determine what light is, further than, that it is something or other capable of exciting vibrations in the ether; for thus it will become so general and comprehensive of other hypotheses as to leave little room for new ones to be invested."—(*Ibid.*)

xii. Many of his views which have been called physical and corpuscular theories, were, I believe, merely modes of expressing what were simple inductive conclusions, or what at least the existing state of knowledge fully justified him in regarding as such.

This, I believe, was the case with the *fits* of easy transmission and reflexion, which were once the subject of so much dispute. What was the plain argument in Newton's mind? he saw that at certain thicknesses *no light appeared on the surface* of the plate: could he then make any other inference than that *none was reflected?* Encountering the lamina then at that particular period of its length, the ray was in some

peculiar state which prevented its being reflected: at the next period it was in an opposite state, and so on. Had he started the idea that there were two portions of light reflected which destroyed each other, in the then state of knowledge it would surely have been justly repudiated as a perfectly chimerical hypothesis.

He adopted, then, this descriptive phraseology simply as expressive of the bare experimental facts. In like manner he referred to the mathematical doctrine of molecular forces, because it enabled him to deduce and compute the laws of refraction and reflexion. Phrases implying material *sides* in a ray, and the analogy of magnetic polarity, he adopted, so far as they sufficed to represent certain classes of facts. The conception of waves in ether, sometimes simply, sometimes jointly with other causes, were referred to when he found them illustrate particular phenomena. But he upheld none of these as generally or even extensively applicable, and was completely alive to the defects and difficulties inherent in each, and even to some objections which have been since removed. He no doubt fully felt that the subject, in its then condition, was wholly unripe for any generalizations: he was therefore content to proceed step by step; to take up this or that hypothesis as a convenient representation of each particular class of phenomena, as they presented themselves, but without the smallest notion of seriously maintaining any one

of them as the true and comprehensive theory, or legitimate mechanical explanation, any further than in the most cautious and general language.

xiii. If to the investigations originally pursued by Newton, of the molecular forces which may produce ordinary reflexion and refraction, we add the more general analytical views of the same doctrine which have since been developed, the introduction of the principle of least action, and the application of it with such profound skill, by Laplace, so as to reduce under the dominion of this theory the laws of double refraction, subsequently extended into so beautiful an analytical system by Malus, (in his prize Memoir, *Theorie de la double Refraction, &c.* 1810,) and adding perhaps the highly ingenious supplementary systems of "oscillations" and "moveable polarizations" of M. Biot, we have named all that can be said to have been done towards erecting the fabric of a theory of molecular emission, applicable to any portion of the phenomena of light.

xiv. It was observed some years since (though perhaps more justly then than it could be now) by a most distinguished philosopher, that if as high mathematical ability had been devoted to the development of the corpuscular theory, as has been to the undulatory, the former might probably have been exhibited in a form equally complete and satisfactory in its application to the phenomena; and this has been

repeatedly appealed to with the highest applause by the anti-undulationists. Now without in the smallest degree questioning the justice of this remark, and without taking into account the different position which the undulatory theory has been able to take, even by the improvements of a very few years past, I would refer not so much to possibilities as to facts—not to what might be effected, but to what has been. Those who lay so much stress on the above quoted remark, like many others who applaud good advice, have done nothing to follow it up. I would go no further than to observe that such application of mathematical recourses *has not been made:* that such a development of theory has not been so much as attempted: that for the great mass of phenomena in fact *no corpuscular theory exists:* and the choice rests between the undulatory theory or none. It is not to be upheld as complete, or unobjectionable, but as the sole explanation; not as being the *best*, but as the *only* theory we have.

But this does not prevent us from being fully sensible of all the imperfections and objections under which the theory labours. The deficiencies in the theory itself attach in several instances even to its fundamental principles. Take for example the question of the *density* of the other; which by some (as by Fresnel) is held to be increased, by others (as Prof. Kelland) diminished within denser media; and by others again, to remain unchanged. The last

opinion has been supported by Prof. Maccullagh and Prof. Challis; from whom however M. Cauchy in his later writings dissents: (*Comptes Rendus*, April 4, 1836, and *Nouveaux Exercic.* liv. 7.) nor can this point be considered as at all satisfactorily cleared up at the present moment. Mr Maccullagh has (*Transactions, Royal Irish Academy*, Vol. XVIII.) upheld his view as essential to that portion of the theory which he has been the first to develop. But it remains to be seen whether it is also reconcilable with the other parts of the system.

XV. The large demands which the theory of waves makes upon our powers of imagination, the moment, at least, we come to combine it with any physical ideas, is not one of its least difficulties to those who are commencing their acquaintance with it. No where is this perhaps more conspicuous than in all that class of enquiries which relate to the mode of propagation of light through space,—its transmission, for example, from the most distant star to the earth.

The fact of the astronomical aberration is most naturally explained by the composition of the motion of the earth with that of a molecule emitted from the star. But on the wave theory (since the propagation from the star *must* be independent of the earth's motion) we are compelled to admit that the ether diffused in space is not displaced by the earth's motion through it, and consequently that it

pervades all substances without any permanent connexion;—"with little or no resistance," as Dr Young expresses it, "as freely perhaps as the wind passes through a grove of trees." (*Experiments on Light, &c. Philosophical Transactions*, 1803.)

This then would seem to involve us, on the other hand, in greater difficulties to account for the refraction and retardation which light undergoes in passing through transparent media. In particular the remarkable fact, so precisely ascertained by M. Arago, (See Biot, *Astron. Phys.* Vol. III. Prof. Lloyd's *Report, B. Assoc.* 1834, p. 313) that the refractive deviation is absolutely unaltered, whatever be the velocity with which the ray impinges on the surface, (determined by observing the prismatic deviation in the light of a star when so situated that the earth was moving directly towards it, and directly from it,) seemed in its consequences equally startling and difficult to reconcile with *any* theory. A new and most complex kind of emission might be imagined to account for it. M. Fresnel, though by a long process, succeeded in reconciling it with the wave theory.

XVI. The difficulties of this question are obviously connected with those just before alluded to respecting the density.

Whether we are to suppose some degree of resistance in the passage of the ether between the

molecules of bodies, or whatever other idea we may adopt, it is clear that we must admit some such cause to operate, as that we may reason upon the case *as if* the whole were stationary, and thus investigate the retardation and refraction of the ray; this last being necessarily in the direct ratio of the former, or the refractive index the reciprocal of the velocity of propagation of the wave.

The general fact of the retardation receives its most direct proof in the well-known experiment of shifting the interference stripes, towards the side on which a thin lamina is interposed: and for palpably demonstrating at once both *its existence* and its *amount*, (as precisely equal to the refraction) I suggested a modification of that experiment, described in the *Philos. Mag.* Jan. 1832, Vol. XI. p. 6, and in my *Elementary Treatise on Optics*, Oxford, 1833.

xvii. In connexion with this point there is also another which appeared to me deserving of more attention than it has usually received. In general, a diminution of velocity may arise either from a diminution in the length of the period in *space*, within the medium, or an increase in the *time* of passing through it, or from both jointly.

Now on the principles of wave *theory*, the *first* of these suppositions must obviously be adopted:

since as each wave within the medium is caused by the swell of a wave incident in the vacuum, every time a wave is produced in the vacuum, there must be also one produced in the medium; or these last must occur at the same times as the former; or the period in *time* of a wave must be constant in all media, and therefore the period in space or the *length* diminished in the denser.

But with respect to the *experimental* evidence to shew whether this hypothesis of diminished *lengths* of period in *space* be the true one, it must be confessed that it rests on a very narrow induction.

Newton, in studying the laws of the coloured rings measured their diameters when formed in *water* between lenses, and found the decrease from those formed in *air* in proportion to the increase in the refractive index; hence he cautiously observes, "Perhaps it may be a general rule that if any other medium more or less dense than water be compressed between the glasses, their intervals caused by interjacent air, are as the sines are which measure the refraction made out of that medium into air." (*Opticks, Obs.* 10, p. 182, 3rd ed.) He nevertheless afterwards positively announces it as a general law in Prop. XVII. (p. 259). He subsequently mentions observing the colours as formed by thin plates of metallic oxides;—(*Obser.* 19), and by films of glass; —(*Obser.* 21, 22) but he states no measures of the

thickness of these films—and when, in the following chapter he gives the table of thicknesses in air, water, and glass, it appears from the observations prefixed, that the values for glass are only calculated from the assumed proportion of the refractive index, supported by the analogy with water.

This appears to have been one of the most remarkable and boldest of Newton's generalizations, and all his successors seem to have adopted it, with as little disposition to recognize any necessity for extending the induction.

It is not less remarkable, that the opponents of the undulatory theory should not have seized the opportunity presented for attack, in a point so confessedly unsupported by experimental determinations, yet so fundamentally connected with the first principle of the theory.

It was in some measure from considerations of this kind, that I suggested a mode of experimentally verifying the conclusion, easily applicable to all media and capable of the greatest accuracy, in a paper in the *Journal of Science*.—(Dec. 1832, Vol. I. p. 435.)

It will however be seen in the sequel, that the application of the theory to the dispersion of light involves indirectly the most decisive means of putting the correctness of this principle to the test.

xviii. The existence of objections arising from experimental results apparently contradictory to the theory, and certainly as yet unrefuted, are obstacles with many, to its reception, more insurmountable than all its intrinsic imperfections. Yet there undoubtedly are several classes of phenomena which the wave-theory has not merely *failed to explain*, but which are apparently at *direct variance*, with its principles.

Of this kind are the objections dependent on photometical measurements, which have been urged with so much acuteness by Mr Potter, founded upon his own experiments, devised and conducted with the highest skill, and investigated with the aid of extensive mathematical knowledge. Some considerations have, it is true, been thrown out tending to modify the application of these calculations.—(See *Journal of Science*, Vol. v. p. 439; XII. 484, and XVI. 16 and 220.) But upon the whole it can hardly be said that the objections have been satisfactorily answered. Whatever opinion may be formed of certain other arguments urged by the same gentleman, or of the tone in which they are brought forward, there can be, I think, but one, of the high value of such objections, for eliciting discussion and thus tending to the final establishment of truth.

xix. Another class of difficulties may be more fairly left for further examination, as belonging to *undetermined questions of fact*.

Such are the *black centered* interference stripes long ago observed by Mr Potter, and which he has recently described further, mentioning certain particular circumstances under which they always appear;—(*Journal of Science*, 1840, XVI. 380,) but, with every attention to the conditions prescribed, I have never been able to produce the effects; accordingly in a communication to the British Association, I strongly urged the necessity for repetitions of these experiments by other observers, that further light may, if possible, be thrown on the causes of apparent failure or success. I regret that no experimenter has hitherto come forward to state the results of such repetition.

XX. Similar remarks will apply to the phenomena of a *dark*-centered diffraction image of an aperture with convex sides, almost touching at the middle, described by Mr Barton, (*Journal of Science*, April, 1833, Vol. II. p. 268). In my discussion with that gentleman, I described my own repetition of the experiment which gave an appearance somewhat similar, though differing in a slight, but material point.

Apart from other considerations, where different observers are at issue as to the *facts*, it is manifestly premature to make an appeal one way, or the other, in discussing the *theory*.

But in the instance last cited there is a further consideration involved, viz. that the theoretical formula

(sufficiently complex even in the case of rectilinear edges) has not (as far as I know) been ever extended to the case of the *curvilinear* aperture in question, it is manifest that the calculation would become immensely laborious. But the case derives a further interest to stimulate enquiry, from the circumstance that something apparently analogous, and not at all explained, is represented as occurring in one of Newton's experiments, (so emphatically described by him as unfinished,) as may be seen in the Opticks, (Book III.) where a dark space appears at C in figure 3, Tab. I.

xxi. Besides these we have the large range of singular and apparently irreducible facts belonging to the *absorption*. *General* principles of reduction of such facts to theory have been most satisfactorily suggested from analogy by Sir J. Herschel; and even precise mathematical investigations, as far as general expressions go, have been given by Professor Lloyd and Mr Tovey, and still further the elaborate investigations of Von Wrede, however complex the analysis, seem not only eminently satisfactory in the general theoretical view afforded, but are applied by the author in detail to a great number of the most remarkable and even apparently irregular cases of absorption. It would be difficult to say why this last named investigation, apparently of a very complete nature in its mathematical details, and involving no other hypothesis than that of successive reflexions of

a portion of the light among the molecules of the medium, (which if not a necessary consequence, is at least a very admissible idea,) should not have attracted that attention or credit to which it seems every way so well entitled. I endeavoured to bring it more into notice by some explanation of it at the Liverpool Meeting of the British Association, and a short abstract of its principle appears in the report of the proceedings. The original memoir is translated in Taylor's *Foreign Scientific Memoirs,* (Vol. I. Pt. 3.)

That such a theory should be at once condemned by some zealous anti-undulationists would not excite surprize. But why it should not have found more favour with the opposite party, may well be a matter of question.

At all events such seems to be the fact, and we must still record the *absorption* as one of the points, on which no explanation, as yet proposed, has met with entire acceptance, even among the supporters of the undulatory theory.

xxii. The defects in fact of the theory, have been freely admitted even by those who have done most for enlarging its boundaries. The following confession of Professor Maccullagh is one which I nevertheless must think overcharged: "It is certain indeed that light is produced by undulations propagated, (however,) with transversal vibrations, through a

highly elastic ether; but the constitution of this ether and the laws of its connexion (if it has any connexion) with the particles of bodies, are utterly unknown. The peculiar mechanism of light is a secret that we have not yet been able to penetrate; as a proof of this, we might observe that some of the simplest and most familiar phenomena have never been explained. Not to mention dispersion, about which so much has been fruitlessly written, we may remark, that the very cause of ordinary refraction, or of the retardation which light undergoes upon entering a transparent medium, is not at all understood. Much less can it be said, that double refraction has been rigorously developed by Fresnel. In short, the whole amount of our knowledge, with regard to the propagation of light, is confined to the *laws* of phenomena, scarcely any approach has been made to a mechanical theory of those "laws."—(*On the Laws of Crystalline Reflexion, &c. Mem. Royal Irish Acad.* Vol. XVIII. p. 38.)

I have given this passage at length, as especially anxious to enter my most serious protest against one expression in it, viz.—the opprobrium of "fruitlessness" which the author has thought proper to cast upon what has been written on the *dispersion*. I trust in the course of the sequel of these remarks, as well as of the body of my tract, to offer ample vindication of the investigations hitherto made from the charge of *unprofitableness*, even if it should be

admitted that they have not yet perfectly established all that can be desired.

xxiii. The leading subject of the following tract is *the dispersion of light.* As introductory to the more detailed discussion of it, it may not be irrelevant here to premise some remarks illustrative of the history of the investigation in its earlier stages, as well as its recent progress.

With regard to the primary *experimental* fact of the unequable dispersion in different media, the truth seems to have found its way very gradually to acceptance through several stages of error. An amount of dispersion in all media, proportional to the mean refraction, was the first idea. But as this was manifestly disproved by a very slight extension of observations, the idea next adopted was that several coloured spaces, though not expanded in proportion to the absolute refraction, were still proportional to one another in different media;—until at length the true view was admitted,—that they follow neither proportion, but some law as yet unknown.

xxiv. The part taken by Newton in reference to the establishment of these results, and the views originally adopted by him, have been much dwelt upon, and may here demand some remark.

Confining his attention to the general constitution of light, it seems to have formed no part of his object to try the properties of any great diversity of media; though on several occasions he mentions incidentally having used such.

Thus in one instance, he employed two different kinds of glass; (*Opticks*, p. 76.) in another, he observes "it is difficult to get glass prisms fit for this purpose, and therefore I sometimes used prismatic vessels, made with pieces of broken looking glasses and filled with rain water; and to increase the refraction, I sometimes impregnated the water strongly with *saccharum saturni.*—(*Ib.* p. 62.) And again he mentions using hollow glass prisms "filled with salt water or clear oil."—(p. 101.)

But more precisely, with respect to the dispersion, speaking of various causes which he shews did not make "any change in the length of the image," (*Spectrum,*) he adds the remarkable assertion, "neither did *the different matter* of the prisms make any; for in a vessel made of polished plates of glass cemented together in the shape of a prism and filled with water, there is the like success of the experiment *according to the quantity of the refraction.*"—(p. 25.) This last expression might seem ambiguous, but we find him on another occasion noticing that with one prism the spectrum was longer than it ought to have been from the refringent angle, and ascribing this to the

greater refractive power.—(See Brewster's *Life of Newton*, p. 50.)

xxv. The difference perhaps might have been nearly proportional to the mean refraction in the few cases he examined*:—But from the turn of Newton's language, it seems certain that it was not *merely* a hasty generalization of this kind, but rather some settled conviction formed on other grounds, which was the real basis of his opinion.

Thus his views respecting the possibility of improving the refracting telescope, which he so emphatically pronounced "desperate", (*Opticks*, p. 91.) clearly imply the belief in an *impossibility* of producing refraction without colour; which yet would have been an obvious consequence of opposed dispersions proportional to each other, but not to the mean refractions.

Still more pointedly was this conviction displayed

* Sir D. Brewster, indeed, in referring to one of these cases, (*Life of Newton*, p. 58,) suggests that the solution of sugar of lead in the water may have made its dispersive power equal to that of glass, but this idea is unsupported by my experiments. I find (*Brit. Assoc. Report*, 1839, p. 11) that the addition of superacetate of lead to water, (a saturated solution) while it increases the mean refraction, causes a *scarcely sensible increase in the dispersion:* the differences of the extreme indices being

for water .... .0131
for solution of superacetate of lead .... .0142
while for the lowest crown glass, it is,...... .0207

in his discussion with Lucas, who using a prism of different glass, found a spectrum much shorter than Newton's; and in the singular positiveness of his remark, "For I *know* that Mr Lucas's observation *cannot* hold when the refracting angle of the prism is full 60°, and the day is clear, and the full length of the colours is measured, and the breadth of the image answers to the sun's diameter," (See *Brewster's Life*, p. 50). We cannot for a moment suppose him referring to a mere surmise which he had taken up prior to an extended induction; his language evidently implies much more: he clearly adopted his belief on grounds to his mind so strong, that not even alleged facts to the contrary could shake it.

xxvi. Most writers have contented themselves with remarking this case as merely an unaccountable instance in which this great philosopher betrayed a singular neglect of his usual inductive caution. But it has appeared to me that some explanation of it at least, may be derived from what appears of the views entertained by Newton as to the actual constitution of the solar spectrum.

Newton always estimated the length of the spectrum by the proportion it bore to its breadth, *when that breadth corresponded to the sun's diameter.* (p. 23.) When this was not the case, he mentions that he often by narrowing the aperture made the length in almost any ratio to the breadth. (p. 59).

Thus the slightest inaccuracy in this adjustment in other observers might easily give different *apparent* lengths. Newton might therefore reasonably look to this as a sufficient source of discrepancy which would disappear if the adjustment were exact.

xxvii. But further, in general, the composition of white light by definite proportions of invariable elementary tints would almost necessarily imply a spectrum always consisting of pure homogeneous coloured spaces, *essentially in the same proportion, if of the same intensity and purity*. Now Newton considered the boundaries of the coloured spaces as absolutely defined, and in the same proportions as the divisions of the monochord. And this constancy of arrangement he traced up to its elements in the series of circular coloured images of the sun; (p. 55) each perfectly homogeneous, distinct, of a constant intensity, and incapable of further condensation or extension of any kind, whose centres were separated in the above proportion. In the spectrum as ordinarily formed these overlapped each other; and thus the colours were in a great degree compound. But he showed that by proper precautions a more complete purity and separation of these elementary images could be obtained. In this point of view then inequality of dispersion would be merely equivalent to a greater or less degree of *purity* in the spectrum. There would be no essentially different property in the unequal lengths of the several spectra;

as they would not be fairly comparable unless of the same degree of purity. Such at least would have been a train of reasoning which would not imply either any unaccountable prejudice, or over-hasty generalization from imperfect data; and, it appears to me, would fully explain Newton's regarding any difference in the prismatic expansion as no more an essential part of the phenomenon than the like difference arising from the *greater or less refringent angle* of the prism: a change in which altered the deviations in proportion to the refraction: and this appears to have been the analogy constantly present to his mind.

xxviii. The question in fact involves that of the intimate nature and constitution of the prismatic spectrum; a subject of which even at this time we have little positive knowledge. If there be a finite number of elementary rays in juxta-position, no further susceptible of decomposition, difference of dispersion can only consist in their ultimate separation by dark intervals of different breadths: and it does not appear that any such separation has ever been effected even when the spectrum is reduced to the utmost purity.

It might perhaps occur to some that we have such intervals in the fixed dark lines: but it seems very unlikely that they are of this nature. It is true they appear broader in more highly dispersive

media; but this is only as seen by the naked eye, and is found to be delusive when they are examined with a telescope,—which shews them continually resolved into groups of finer lines. And if the bright intervals were only repeated images of the slit, they ought to be all equal to it, and to each other in angular breadth; or, in other words, the dark lines should be arranged at equal distances uniformly through the spectrum; both which are contrary to observation: higher magnifying powers continually bring out new lines; and thus reduce the bright spaces to breadths successively smaller. And I may here observe this last subject is one which has by no means been completely examined. I am not aware of the power of Fraunhofer's telescope, but it could not have been great from the size. It would be highly interesting if those who possess powerful achromatic telescopes would examine minutely and describe the appearance of the lines in the spectrum as shewn by them.

Again, can it be said that any given ray is more pure in a highly dispersive medium than in a lower? or in what respect do the corresponding rays really differ? To such questions I believe we are not at present in a condition to give any decisive answer. I conceive it still remains to be explained, in what precise respect the difference in the analysis of white light by prisms of different media is consistent with the invariability of the result of synthesis.

xxix. These questions however are now recognized as distinct in their nature from those which belong to the study of the law of dispersion in different media. Here the main object has been to examine simply the *measurable phenomena*, as presented to our observation, in comparing the spectra formed by different media: to distinguish as well the variations in the *total* amount of elongation in those spectra, as in the *relative* expansions of the particular coloured spaces.

Both these enquiries were beset with difficulties: but a considerable series of the general facts had been classified with his accustomed acuteness by Dr Wollaston; and subsequently determinate measures of the *amount* of dispersion obtained through an immense range of refracting media by Sir D. Brewster. Yet no precise measures for *definite rays* were given till Fraunhofer employed for this purpose the fixed dark lines in the polar spectrum originally discovered by Dr Wollaston; but as originally observed,—enlarged in number from 6 to nearly 600,—and for the first time accurately measured by himself.

xxx. As to the *law* of the dispersion, though the first notion of a simple proportionality had been disproved, no other was substituted. The refrangibility was seen to vary considerably and apparently irregularly for each ray and each medium, and when

Fraunhofer had assigned exact numerical values as the varying refractive indices throughout the several spectra, the apparent absence of any law connecting these numbers was only rendered more palpable. All that could be said was that the numbers observed a regular increase from the red to the blue end of the scale, but in different ratios in each medium.

The first object of inquiry in the search after such a law would be some other characteristic of the same definite rays equally well determined, between which and the refractive index some connexion might possibly be found to exist.

The only such characteristic is perhaps the length of the wave, which is manifestly a real object of experimental measurement by whatever name it may be called. And this important datum was also supplied by the refined and precise observations of Fraunhofer, on the perfectly pure spectra formed without any prism, simply by interference, seen on looking at a narrow slit, or line of light, through a telescope whose object glass is covered with a fine grating of parallel threads, parallel also to the slit. These spectra under very favourable circumstances exhibit the same dark lines, whose positions depend solely on the wave-lengths; and their distances from the center are in the simple ratio of those wave-lengths which were thus most accurately determined by Fraunhofer from the observed distances.

xxxi. The first, and I believe only attempt to discover a law connecting the index and wave-length, by induction simply and comparison of observations, was that made by M. Rudberg, (See *Phil. Mag.* 1827, Vol. II. p. 401; *Ann. de Chimie,* Tom. XXXVI, XXXVII) in which he gives a formula expressing such a relation as a very simple function; which he verifies with much accuracy in all the indices determined by Fraunhofer. I call this a *purely inductive* or *empirical law,* having given my reasons for believing that the author himself so regarded it in a paper in the same journal. (*Phil. Mag.* and *Journal of Science,* Feb. 1836).

xxxii. When the attempt was made to advance to any *theoretical* principles which should assign such a relation between the wave-length and the index, a very little study of the undulatory hypothesis, in its common acceptation, showed apparently insuperable difficulties. Upon received principles, though it readily explained the refraction of *homogeneous* light, yet it directly contradicted the idea of an unequal refragibility of rays with different wave-lengths.

These difficulties were indeed triumphantly held forth by the opponents of the undulatory theory as absolutely fatal to its claims; but they are by no means *peculiar* to this theory. The hypothesis of emission has not been at all more successful in affording any satisfactory explanation.

Let us, however, look at the nature of the difficulty as it occurs upon the ordinary hypothesis of undulations. The front of a wave, incident obliquely on the surface of a transparent medium, and arriving successively, *e.g.* at any two points of the surface, at each originates a new spherical wave within the medium. If the refractive power be greater, these are propagated with diminished velocity. The second of these new waves within the medium has propagated itself a little way before the first has gone through the same space as the original wave in the same time. Hence the plane touching their contemporaneous surfaces will be inclined to the surface of the medium at a less angle than the front of the original wave; and (it is easily seen) precisely so much so, as that the ratio of the sines is that of the velocities, or is equal to the index of refraction.

The refraction, then, depends *solely* on the diminished *velocity* of propagation of the waves, and ought to be exactly the *same for waves of all lengths,* unless there could be shown any *connexion* between the *length* of a wave and the *velocity* of its propagation.

The essential point aimed at in any legitimate inquiry of this nature is to show some relation between the *length* of an undulation and the *velocity* of its propagation; or in other words, that in trans-

parent media the velocity of propagation of the waves is different for the different primary rays, that is, for rays in which the lengths of the undulations are different.

But, as we have seen, in the ordinary view of the theory of waves the *equal* refrangibility of all rays is a necessary consequence. The course, then, to be pursued by any judicious inquirer, and that, in fact, adopted by M. Cauchy, is that of reviewing the first elements of the theory, viz., the particular constitution of the hypothetical ætherial medium, and endeavouring so to modify them, that while they shall apply equally to the conclusions deduced on the ordinary principles, and referring to the other phænomena of light, they shall also be made to include results which will explain the phænomena in question. Now, the great desideratum, the establishment of a relation between the length of a wave and the velocity of its propagation, is supplied, as M. Cauchy has expressly remarked, "*in general*," by his investigation.

xxxiii. The primary object in a mechanical theory is to connect the simple mathematical representation of waves with acknowledged principles derived from dynamics. Such principles were in some degree established by Huyghens, and the more precise dynamical theory of vibrations of an elastic medium, as in the case of sound, investigated by

Euler, and D'Alembert (*Berlin Acts*, 1747), and by Lagrange (*Turin Memoirs*, 1759), was soon seen to apply (at least generally) to the analogous case of undulations of æther, constituting *homogeneous* light. But with reference to many peculiar properties, as of *heterogeneous* light, and even in the instance of the propagation of *transverse* vibrations, (which Young had shown to be a necessary supposition) further modifications and extensions of the theory were required. Such extension in any great degree of generality was first made by Fresnel, principally in his Memoir on Double Refraction (*Mem. Acad.* Vol. VII.) who clearly pointing out the too limited nature of the ideas of æther hitherto adopted, laid down that of *molecules separated by finite intervals*, by which the whole system of transverse vibrations, of polarization, and the wave-surface, were established by an analysis which must ever remain a monument of the profound and comprehensive genius of its author.

But the mechanical principles are expounded chiefly in general terms, and the results are not deduced from a mathematical development of equations of motion. Thus these researches, though of the highest order, were soon found not to possess in all points that comprehensiveness of principle which would satisfy the increasing demands of discovery; and in some instances the theory was confessedly left defective.

xxxiv. Meanwhile investigations, apparently of a very distinct and purely abstract nature, were being carried on by mathematicians respecting the motions of an assemblage of molecules so constituted as to form an elastic system. From these were derived more comprehensive dynamical formulas, which were at length applied to the phenomena of light, so as not only to embrace and generalize the former theories, and to establish them on a more connected mathematical basis, but also to include the explanation of other phenomena, which the former methods had failed to furnish.

The first who pursued such dynamical speculations appears beyond question to have been M. Navier, in a paper read to the Academy of Science, May 14, 1821, and published in its Memoirs, Vol. VII. p. 375, (1827); entitled "Mémoir sur les lois d' l'equilibre et du mouvement des corps solides elastiques." The principles however, on which the enquiry is conducted, apply to elastic systems in general: and he deduces equations for the motions of particles slightly disturbed in three rectangular directions, when the elasticity is the same in every direction. In this there is no specific reference to æther or to light. But the equations in question are identical for this particular case, with those afterwards deduced by M. Cauchy from similar principles, as applied to an elastic system in its most general constitution; whence he subsequently deduces expressions for the propa-

gation of waves. These equations may therefore be regarded as containing the fundamental dynamical principles of the theory of light.

The abstract portions of these researches of M. Cauchy appeared in several successive papers, interspersed with others in the 3rd and 4th volumes of the *Exercices de Mathematiques*, (1827, 1828.) The particular equations in question are deduced in Vol. III. p. 191, *Eq.* (16, 17), and the author points out their accordance with those of M. Navier in the same Vol. p. 212. The more specific application to light was made in the 5th Vol. (1829), especially in livrasions 50 and 51. Here the publication terminated abruptly.

XXXV. My present object restricts my remarks to those parts of M. Cauchy's writings which bear directly on the dispersion. I therefore do not here advert to the various discussions on other points in the theory in which M. M. Poisson, Lamé, and others have borne a part. The death of M. Poisson, in every respect so irreparable a loss to the scientific world, has been in no respect more so than in the circumstance of his having left unfinished his work on the theory of light, which would probably have supplied many of the desiderata in some of the most interesting branches of the subject.

To return then; in following out certain points of his system more closely, M. Cauchy found the

principle of its application to the long sought solution of the problem of dispersion. This was made known by the publication of his "*Mémoir sur la dispersion de la lumiére*," in 1830, as a separate tract. This memoir contains the entire investigation, from the first principles as far as relates to the general theory of dispersion, but it does not contain any explicit deduction of a formula expressing the law of the unequal refrangibility. However, in his memoir read to the Academy of Sciences in June, 1830, M. Cauchy made such a deduction, though only in an approximate form, and published it in a paper in *Ferussac's Bulletin*, 1830, Tom. XIV. p. 9.

xxxvi. These researches however failed to attract attention in England till some years afterwards. The earliest published notice of them which appeared in this country, was given in Sir D. Brewster's Report on Optics to the British Association, at the Oxford Meeting, 1832. In the complete and masterly report of Professor Lloyd, on the same subject, addressed to the Association in 1834, the able analysis given of M. Cauchy's investigations, more especially in their general relations to polarization, the wave-surface, &c., includes but a very slight reference to the question of the dispersion.

xxxvii. My attention having been drawn to the subject, more especially by the suggestions of Mr

Airy, I had, in 1834, translated and abridged the memoir of M. Cauchy, not aware at the time that the author had deduced a formula for the dispersion. Such a formula however, independently deduced, and an important condition in its application, suggested by Mr Airy (in a letter to myself, June, 1834,) were stated and explained by me, at the Edinburgh Meeting of the British Association, Sept. 1834. My abstract of M. Cauchy's researches appeared in the *Journal of Science*, Vol. VI. commencing Jan. 1835. And in the No. for April, 1835, it was followed by an appendix containing the deduction of the dispersion formula, together with additional investigations; some of which were contributed by Sir W. R. Hamilton, for developing the formula into series, and facilitating its application to the numerical calculations by which theory was to be compared with observation. In the course of these papers I referred to the law of M. Rudberg, and shewed (March, 1836.) that it was not difficult to see how that formula might accord nearly with the approximate formula of theory, under certain very restricted conditions.

But it must be borne in mind, with respect both to this and all the *earlier* of the subsequent investigations, that they extend only to a range of results by no means *high* in the scale of dispersion, and it is obvious that any law could receive its full verification only from the higher cases. No

such cases however, at that time, had been investigated or exhibited in the form of a series of indices for definite rays.

xxxviii. I proceeded, however, to the work of verifying the theory by a comparison with such observed results as were at that time known to me.

In February, 1835, I had completed the calculation of all the indices observed by Fraunhofer, by means of the formula deduced from M. Cauchy's theory, (before referred to) though only in an approximate form. The coincidences are allowed, I believe, to be as close as can be expected. The paper was read to the Royal Society on the 12th of March, and printed in the first part of the *Philos. Trans.* for that year, entitled "*Researches towards establishing a theory of the dispersion of light.*" Meanwhile the investigations of M. Cauchy were continued under the title of "*Nouveaux Exercices,*" &c.; the first four livraisons appeared successively during the year 1835, at Prague. The first is a repetition of the *Memoir sur la dispersion.* In the second, (besides other discussions) is contained the actual deduction of the formula for dispersion. In September, 1835, the same author circulated a lithographed memoir on interpolation, (a translation of which appears in the *Journal of Science,* Vol. VIII. p. 459.) in which he gives a calculation by that method, of Fraunhofer's

indices for one kind of flint glass, but without any theoretical explanation. In the "*Nouveaux Exercices*" for 1836, he gives in detail his most elaborate and exact method of computation, and applies it with perfect success to all the indices determined by Fraunhofer: noticing the approximate formula which I had used, and which is the same in substance as that he had before published, (p. 234.) Before this however, having become acquainted with another series of indices determined by M. Rudberg in certain crystals, I compared all these with theory by the same approximate method as that used in my former paper, and the accordances were, I believe, generally allowed to be as close as the former. The results were communicated to the Royal Society in Nov. 1835, and published in the first part of the *Philos. Trans.* for 1836, under the title of "*Researches, &c.* No. II."

xxxix. In these instances the work of determining the indices was done to my hands, and I could proceed to the theoretical computations with the most perfect confidence in the accuracy of experimental data, furnished from the labours of observers so well known for precision and skill, and obtained, too, before the formula of theory had been deduced.

In any comparison of theory with experiment, it is, in all points of view, far more satisfactory that such comparison should be made with the observa-

tions of others rather than those of the theoretical computer himself.

In the researches which I have carried on subsequently to those just alluded to, however, this desirable condition has not been fulfilled. Though the importance of obtaining a series of indices for the standard rays in different media had been long since pointed out and acknowledged by the most eminent philosophers, yet no observer was found to undertake the task of carrying on the work which Fraunhofer and Rudberg had so successfully begun. I was thus left to make an attempt myself to supply the deficiency for various other media, including those of the most highly dispersive nature. My first results were communicated to the British Association at the Meeting, 1836, and are published among the memoirs of the Oxford Ashmolean Society. The observations were reprinted and corrected, and other results given in some supplementary papers in the same collection. They have been since finally compared and some new results added, in my "*Report on Refractive Indices*," read to the British Association, and printed in the volume for 1839.

From the remarks prefixed to those results, the scientific reader will, I trust, be sufficiently enabled to judge of the nature and degree of accuracy of the observations, as well as of the great and unexpected amount of the difficulties attending the prosecution

of them. Such as they are, however, these results form (as far as I am aware) the only existing data for pursuing the comparison with theory. But I trust they may not be thought insufficient, when we consider that in the present stage of the enquiry, the object to be aimed at seems chiefly such a general comparison, as may enable us to see whether the main principle of the undulatory explanation of the dispersion be applicable, with a sufficient approach to precision, to encourage us to pursue the theory, or whether it must be abandoned, and some new principle sought. It will, however, be one main object in a future section to review in detail this part of the enquiry, and the progress made in it: the more so, as I believe much misconception exists respecting it.

xl. Among the contents of the series of papers which I published in the *Journal of Science*, already referred to, the most valuable portion consisted of some investigations communicated by Sir W. R. Hamilton, these related chiefly to the deduction of a formula suited to calculation, derived from the *exact* expression of theory, without any of those modified suppositions, which rendered that before employed only approximate. This formula possesses the material advantage, of making the calculations far shorter and easier in practice than even the approximate method before used.

It was by this formula, that in Oct. 1836, I compared with theory all the most material cases in

the series of observations last alluded to, including the most highly dispersive media yet examined. The results were published in the *Philos. Trans.* 1837, Part I., entitled "*Researches, &c.* No. III."

xli. Mr Kelland's paper (read Feb. 22, 1836,) in the *Cambridge Transactions*, Vol. VI. contains an original investigation of the same general theory, commenced without a knowledge of M. Cauchy's researches, greatly simplifying the equations and leading to a formula for calculation, by which the author computes all the indices of Fraunhofer; in which the accordances are extremely close; though with the utmost fairness he notices expressly certain small anomalies which present themselves. The continuation of this paper includes many other highly important investigations, but less directly connected to our present subject. By this method I recalculated three of the most important cases of my former series, in "*Researches, &c.* No. IV." inserted in the *Phil. Trans.* 1838, Part I.

Of all these results I shall say nothing here, reserving my remarks to their proper place in the body of this essay.

Mr Tovey, in January 1836, commenced the publication of a series of researches on the Undulatory Theory (*Journal of Science*, Vol. VIII); in which he adopts greatly simplified analytical methods to

deduce the same general conclusions with respect to the dispersion as those independently obtained in the papers just referred to. But subsequently, (besides some other valuable investigations) in the same journal (Vol. XII. Jan. 1838) this author produced a short paper of a more entirely original character, and disclosing an apparently new element in the theory of vibrations, viz. the existence of a *relation* between the peculiar supposed *arrangement of the molecules* of the ætherial medium in space, and the *nature of the vibration as elliptic or rectilinear*. The character of such arrangement is indicated mathematically by the presence or evanescence of certain terms in the differential equations. These terms had been regarded as *evanescent* by M. Cauchy and others: and this as a consequence derived from the hypothesis of a *uniform* distribution of the ætherial molecules in space. Mr Tovey's object in the paper referred to is to shew that when this is *not the case*, elliptic polarization is the result.

It has been alleged indeed that there is some reference to such a principle in M. Fresnel's researches (*Mém. Sur la Double Refraction*), but I think it distinctly appears on examination that his view is restricted to certain arrangements of the molecules in crystalized media, and after all it is unconnected with differential equations of motion. This point, alluded to in a hasty note to my paper in the *Journal of Science*, March, 1841, is, I con-

ceive, set at rest by a short supplement inserted in the following number of that Journal.

xlii. Convinced of the high value of this investigation, but conceiving that the remarkable conclusion was not so fully explained as its importance seemed to render desirable, in a paper inserted in the *Phil. Trans.* for 1838, Part II., I endeavoured to establish and further elucidate the conclusion by what seemed to me a more direct and perspicuous method.

The intimate connexion between these theoretical views and the important points discussed in several masterly papers of Sir J. Lubbock, (*Journal of Science*, Vol. XI. 1837, and XV. 1839), was soon rendered evident. The direct object of these papers was chiefly the illustration of Fresnel's views respecting the axes of elasticity and the wave-surface; and there appeared at first sight some degree of difficulty as to the connexion between these deductions and the views just referred to. I endeavoured to draw attention to the subject in a short communication to the British Association at Birmingham, 1839.

In consequence of some correspondence and further discussion, I drew up a supplement to my last-named paper, which was inserted in the *Phil. Trans.* 1840, Part I., in which I pointed out the modifications which the former conclusions would receive in connexion with the principles thus elucidated.

This supplement, however, was very brief: other points also appeared to call for further notice, and the whole investigation seemed to require re-casting. Accordingly, in a paper inserted in the *Journal of Science*, (March, 1841, Vol. XVIII. No. 116), I gave the whole investigation in such a revised form, as I trusted might free it from the ambiguity and doubt in which it seems to have been involved.

The substance of that investigation will be included in the following Tract. It is in fact involved in the process by which the dispersion formula is here deduced. To this main subject (as before observed) the treatise is primarily devoted: but the discussion of it necessarily leads me to touch upon some other topics related to it, into which however I do not pretend to enter in detail: I merely point out their connexion, and refer the student to sources of full information*.

xliii. There is indeed a great want at present of a *complete* supplement to the existing treatises, such as those of Mr Airy and Sir J. Herschel, (notwithstanding the valuable additions annexed by M. Quetelet to his translation of the last-named work,)

* Among such sources the Memoirs of Fresnel and of Cauchy, (*Mem. Inst.* x.) above referred to, I am happy to learn are likely soon to be made fully accessible to the English reader by translations in the valuable series of Foreign Scientific Memoirs, publishing by Mr Taylor.

which shall embrace all the investigations carried on since the date of those publications.

The development of certain parts of the theory is placed within the reach of the student, in the valuable papers of Prof. Sylvester "on the optical theory of crystals, &c." in the *Journal of Science*, Vol. XI., XII. which it is much to be wished were republished in a separate form; while its already high value would be greatly enhanced by somewhat more of explanatory remark: especially as regards its connexion with other investigations above referred to.

xliv. In the wish to render this essay as generally useful as possible, I have taken up the subject from its elementary principles; believing that among the hindrances to the general reception of the theory none is more injurious than a prevalent, but most erroneous idea, that it is of a very difficult and abstruse nature. If acquainted only with the very first principles of the integral calculus and analytical dynamics, the reader will be stopped by no mathematical difficulty in the following pages.

For the more complete review of the whole subject, I have availed myself of the illustration derived from brief abstracts of the investigations of several of the authors already alluded to. And in acknowledging obligations of this kind, I feel bound particularly to name Sir J. W. Lubbock, to

whose permission to make use of his valuable materials, as well as to his advice and assistance generally, my work is largely indebted.

xlv. It is with the same object of general utility in view, that I have introduced as an appropriate illustration of this volume, a chart in which the spectra produced by prisms of the same angle, of different media, are laid down to a scale, so as to exhibit to the eye the absolute and relative deviations of the different primary rays in each. It requires no further explanation.

xlvi. For the sake of those readers also, who may be commencing their acquaintance with the undulatory theory, it may not be out of place here to mention a method of imitating the different kinds of vibrations producing a wave by mechanical means. A contrivance of this kind, exhibiting the nature of plane polarized, and circularly polarized light, was made some years ago by Mr Airy. I subsequently devised a similar machine, which shews all kinds of elliptic vibrations, from circular to rectilinear, by a simpler mechanism. It would be difficult here to describe it satisfactorily, but the essential principle is that of a simple crank, having a rod which passes through a ring fixed at a certain height above the center of the circular motion. If this height be great compared with the radius, the extremity of the rod (marked to the eye by a white ball) will move

up and down in what is to sense a straight line, if however the ring be lowered, then the course described by the ball, will become more and more *oval:* or to the eye will represent *elliptic*, and at length, *circular* vibrations. The curves traced out, in fact, are not mathematically elliptic, but ovals of a high order. It is difficult to deduce their equation;—nor have I met with any account of them;—they would furnish an interesting problem. To return however;—a single vibration being thus produced, it only remains to have a number of such rods, with cranks, to turn on one common axis: but each crank so arranged that the extremities of them, or points of junction with the rods, lie in a helix round the axis. It is then obvious that a wave is produced among the balls and will be propagated either way, as the handle is turned, with vibrations rectilinear, or oval of various degrees, according to the height of the bar supporting the rings*. The conception of a wave is often found difficult by learners, but I believe only in consequence of the confused notions we are apt to derive from contemplating the production of it in the common instances of its occurrence. A glance at an illustration of this kind enables the mind to grasp the idea at once, even when unaccustomed to the mathematical analysis of it.

* Such a machine was deposited in the Gallery of Practical Science, Adelaide Street, and similar ones are made by Mr E. M. Clarke of the Strand.

SECTION I.

ELEMENTARY PRELIMINARIES.

It may be convenient to the student to commence by briefly recapitulating, in what will be little more than a sort of Syllabus, the principal elementary formulas which contain the analytical view of undulations.

WAVE-FUNCTION.

1. The whole conception of a *wave*, in a mathematical point of view, is *included* in the analytical formula by which it is expressed. In other words the definition of a *wave* is the aggregate of the motions represented by a function of the following kind:

Let the distances of successive molecules at rest, in a straight line from the origin be x, $x + \Delta x$, &c., and the times t, $t + \Delta t$, &c. from a fixed epoch. Let all these molecules perform vibrations in the same time τ.

2. Then for any molecule at a distance x, to represent the phase of vibration at the time t, if we take such a function of that time, and of the distance, as to fulfil the condition of *periodicity*, that is, in other words, such as to give always,

$$\phi\,(t,\ x) = \phi\,\{(t + \Delta t),\quad (x + \Delta x)\}.$$

We shall express a *wave* propagated in the direction $+ x$, in the same time τ as that of a vibration, and whose length between points of like phase we call λ.

3. If it be propagated with uniform velocity (v), we have also

$$\Delta x = v \Delta t,$$

$$\text{and } \lambda = v\tau.$$

4. If we substitute two quantities n and k, such that we have $v = \frac{n}{k}$, then from (3) we have

$$k \Delta x = n \Delta t.$$

5. And introducing these quantities in the form (2), it is easily seen that the function in question can only be

$$\phi \{n (t + \Delta t) - k (x + \Delta x)\}$$
$$= \phi (nt - kx).$$

6. If we had supposed the motion in the direction $-x$, this would become

$$\phi (nt + kx).$$

Or in general, to express waves propagated both ways from a center of disturbance,

$$\phi (nt - kx) + \phi (nt + kx).$$

7. The conditions of such a function are fulfilled in the most obvious manner by the simple trigonometrical forms, as $\sin (nt - kx)$ or $\cos (nt - kx)$, &c.

8. The extreme distance to which a molecule recedes from its point of rest, or amplitude of the vibration being a, the displacement u is thus expressed by

$$u = a \sin (nt - kx) + a' \sin (nt + kx).$$

9. But as we usually examine only waves propagated in one direction, we take,

$$u = a \sin (nt - kx).$$

10. A constant added is merely equivalent to a change in the origin, or period of commencing the vibration, as

$$u = a \sin(nt - kx \pm b).$$

11. The sum of a number of functions of the same form represented by

$$u = \Sigma a \sin(nt - kx)$$

expresses the assemblage of waves which constitutes a sensible ray of light.

12. The retardation of a wave, and consequent refraction of a ray within a medium, (expressing the refractive index by μ) gives the relation

$$v = \frac{n}{k} = \frac{1}{\mu}.$$

13. Also in the assumption (4) we may have consistently with (3)

$$k = \frac{2\pi}{\lambda}, \quad n = \frac{2\pi}{\tau},$$

when λ is the wave length within the medium.

14. From one medium to another, the velocity of the propagation of waves is diminished in the ratio of the refractive indices. But the number of waves passing in a given time through the same point will be the same, and consequently the time (τ) of a vibration (which is also that of an undulation) will be constant for all media. Thus we have for the same ray in any medium

$$\frac{1}{\mu} = \frac{\lambda}{\tau}, \quad \text{or} \quad \mu\lambda = \tau.$$

15. Hence if for any ray the wave length in air be λ, and in any medium λ' we have

$$\frac{\lambda'}{\lambda} = \frac{1}{\mu}, \quad \text{or } \lambda = \mu\lambda'.$$

16. In employing functions of this kind we have often occasion to consider their increments, the expressions for which it may be convenient to premise as follows:

In the function,

$$u = \sin(nt - kx)$$

we have the increment,

$$\Delta u = (\cos k\Delta x - 1)\sin(nt - kx) - \cos(nt - kx)\sin k\Delta x$$

$$= -2\sin^2\frac{k\Delta x}{2}\sin(nt - kx) - \sin k\Delta x\cos(nt - kx) = D.$$

17. Again for,

$$u = \cos(nt - kx)$$

$$\Delta u = -2\sin^2\frac{k\Delta x}{2}\cos(nt - kx) + \sin k\Delta x\sin(nt - kx) = D'.$$

18. Similarly for,

$$u = \sin(nt - kx \pm b),$$

$$\Delta u = D\cos b \pm D'\sin b.$$

19. And for,

$$u = \cos(nt - kx \pm b),$$

$$\Delta u = D'\cos b \mp D\sin b.$$

20. And in all these cases the increment is generally of the form,

$$\Delta u = M\sin k\Delta x + N\sin^2\frac{k\Delta x}{2}.$$

21. We may also remark that either of the above expressions may be written,

$$\Delta u = -2 \sin^2 \frac{k \Delta x}{2} u - \sin k \Delta x \frac{du}{dx}.$$

22. Again in this (or in the preceding formulas) some writers prefer employing the expression,

$$\Delta u = -\operatorname{versin} k \Delta x\, u - \sin k \Delta x \frac{du}{dx}.$$

INTEGRATIONS.

23. These wave-functions in their most general form are solutions of the partial differential equation,

$$\frac{d^2 u}{dt^2} = -v^2 \frac{d^2 u}{dx^2}.$$

provided v be constant, or provided n and k be constant, or their ratio $\frac{n}{k}$ constant.

In the case of the more simple trigonometrical functions, the solution is manifest at once from the obvious property of such functions to reproduce themselves in alternate successive differentiation.

24. If u involve only one variable t or x, we thus have

$$\frac{d^2 u}{dt^2} = -n^2 u, \text{ \&c. } \frac{d^2 u}{dx^2} = -k^2 u, \text{ \&c.}$$

25. If u involve both t and x the partial differential equation results

$$\frac{d^2 u}{dt^2} = -\frac{n^2}{k^2} \frac{d^2 u}{dx^2}, \text{ \&c.}$$

26. In a case nearly allied to this, and in a similar way, a solution of the partial differential equation

$$\frac{d^2u}{dt^2} = n^2\left\{\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right\}$$

is easily seen to be

$$u = \sin\{nt - (ex + fy + gz)\},$$

provided

$$e^2 + f^2 + g^2 = 1.$$

27. Another case of a somewhat similar kind has been considered by M. Poisson, and since by other writers, to which we shall also have occasion to refer, in the following manner:

Any function of t and x may be expressed by a series of terms each of which is of the form

$$u = p \sin kx + q \cos kx$$

where p and q are functions of t, and k a constant. In like manner any function of t may be expressed by

$$v = a \sin nt + \beta \cos nt,$$

whence on substituting such functions for p and q we find

$$u = \begin{cases}(a \sin nt + \beta \cos nt) \sin kx \\ + (a' \sin nt + \beta' \cos nt) \cos kx\end{cases}$$

which again may be put in the form

$$u = A \sin(nt - kx + b) + A' \sin(nt + kx + b').$$

28. On this principle the solution of the partial differential equation,

$$\frac{d^2u}{dt^2} = h_1 \frac{d^2u}{dx^2} + h_2 \frac{d^4u}{dx^4} + \text{\&c.}$$

will be found to be

$$u = p \sin kx + q \cos kx.$$

where k is a constant and p and q functions of t. For on forming the partial differential coefficients, we find

$$\frac{d^2u}{dt^2} = \frac{d^2p}{dt^2}\sin kx + \frac{d^2q}{dt^2}\cos kx,$$

$$\frac{d^2u}{dx^2} = -pk^2 \sin kx - qk^2 \cos kx,$$

$$\frac{d^4u}{dx^4} = pk^4 \sin kx + qk^4 \cos kx, \text{ \&c.},$$

and on substituting these values the equation (28) will give

$$\left.\begin{array}{l} \left\{\dfrac{d^2p}{dt^2} + (h_1k^2 - h_2k^4 + \&c.)\,p\right\}\sin kx \\ + \left\{\dfrac{d^2q}{dt^2} + (h_1k^2 - h_2k^4 + \&c.)\,q\right\}\cos kx \end{array}\right\} = 0.$$

29. And since this must hold good for all values of x we must have the coefficients of $\sin kx$ and of $\cos kx$ severally $= 0$, or writing for abridgment

$$h_1k^2 - h_2k^4 + \&c. = n^2,$$

$$\frac{d^2p}{dt^2} = -n^2p,$$

$$\frac{d^2q}{dt^2} = -n^2q,$$

which are of the same forms as (24); or we have

$$p = a \sin nt + \beta \cos nt,$$

$$q = a' \sin nt + \beta' \cos nt,$$

or as above, (27),

$$u = a \sin (nt - kx + b) + a' \sin (nt + kx + c).$$

VIBRATIONS REFERRED TO THREE CO-ORDINATES.

30. Looking to the displacements of any one molecule ξ, η, ζ are the rectangular components of a resultant displacement ρ; and if r be the distance from the origin, a molecule first disturbed in the direction of the ray {confining ourselves to the waves propagated on the positive side as above remarked (6,)} we shall have,

$$\xi = \Sigma \{a \sin(nt - k\mathrm{r})\},$$
$$\eta = \Sigma \{a' \sin(nt - k\mathrm{r})\},$$
$$\zeta = \Sigma \{a'' \sin(nt - k\mathrm{r})\};$$

and in like manner,

$$\rho = \Sigma \{A \sin(nt - k\mathrm{r})\}.$$

31. If X, Y, Z be the angles which ρ makes with the semiaxes, we have also,

$$\Delta\rho^2 = \Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2,$$
$$\Delta\xi = \Delta\rho . \cos X, \quad \Delta\rho = \frac{\Delta\xi}{\cos X}, \quad \cos X = \frac{\Delta\xi}{\Delta\rho},$$
$$\Delta\eta = \Delta\rho \cos Y, \quad \&c.$$
$$\Delta\zeta = \Delta\rho \cos Z, \quad \&c.$$
$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1.$$

32. Another expression, sometimes referred to, is found by the identical form,

$$\Delta\rho = \Delta\rho\,(\cos^2 X + \cos^2 Y + \cos^2 Z),$$
$$= \Delta\xi \cos X + \Delta\eta \cos Y + \Delta\zeta \cos Z,$$

whence we have

$$\rho = \xi \cos X + \eta \cos Y + \zeta \cos Z.$$

33. We have spoken of the *ray* as propagated *in a certain direction;* that is, conceiving a centre or origin of disturbance, the vibratory movement will be communicated to molecules at successive distances in the man-

ner above explained, on every side of that centre:—whether with equal rapidity, in all directions, will depend on circumstances not yet adverted to: but supposing (for the sake of illustration) that the waves spread on all sides round the origin in spherical shells, or indeed in curved surfaces of any form, we may always fix upon any one radius, or direction, from the centre in which to investigate the mode of propagation of the motion and the laws which regulate it.

The oscillations which each molecule performs and whose aggregate in successive phases constitutes the wave, have been considered only agreeably to the general illustration above proposed, and nothing has been as yet assumed with regard to the directions in which they are performed as referred to the direction of the ray; they may coincide with it, or be in any way inclined to it.

34. For all ordinary phenomena of interference, &c. it is immaterial in what direction we suppose them to take place. But the interferences of polarized light and other phenomena connected with them, require us to suppose *the vibrations wholly transverse to the ray, and generally in planes perpendicular to it.*

Thus, if we take the ray as coinciding with the axis x, then on the principle of transverse vibrations, we have

$$\xi = 0,$$
$$\eta = \Sigma \{a \sin (nt - kx)\},$$
$$\zeta = \Sigma \{\beta \sin (nt - kx)\}.$$

POLARIZATION.

35. By such formulas under appropriate conditions, the several cases of unpolarized and polarized light are expressed;—

In common or unpolarized light the vibrations are in all possible azimuths round x; hence the coefficients α and β are wholly arbitrary and independent.

In plane polarized light we have, on Fresnel's principles, (supposing the plane of polarization to coincide with that of y or z, which we may always do without loss of generality)

$$\alpha = A \cos i, \qquad \beta = A \sin i,$$

where i is the angle formed by the plane of vibration with another plane, which he terms the plane of polarization: also the squares of the amplitudes expressing the intensities of light,

$$\alpha^2 + \beta^2 = A^2.$$

The formulas in this case represent *two* rays polarized in planes at right angles.

If we consider only one ray wholly polarized in either plane, it is equivalent to supposing either $i = 0$, or $i = \frac{\pi}{2}$, or that one of the formulas disappears.

36. M. Cauchy (in his earlier writings,) Professor Maccullagh, Mr Tovey, and other mathematicians, term the plane of polarization that *in* which the vibration is performed. M. Fresnel uses the same term to signify the plane *perpendicular* to this. This difference in terms, however, involves consequences which affect the subsequent applications of the theory.

In other words, the difference is not merely one of terms, as it might appear, but implies this question, whether the vibrations are performed *in* the plane of reflexion or refraction (as the case may be) or *perpendicular* to it. The explanation of the ordinary phenomena of reflexion, refraction, and interference, are independent of this distinc-

tion: they are even independent of the consideration whether the vibrations are in the direction of the ray or transverse to it: but as when we come to the interferences of polarized light we are obliged to take this last point into account, and find it necessary to adopt the transverse direction, so when we arrive at certain points connected with double refraction, the wave-surface, and with the laws of crystalline reflexion and refraction, then the further condition of the plane of vibration must be introduced.

This distinction then, will not affect our immediate investigation.

M. Cauchy in his paper on the Theory of Light, (*Mem. Instit.* Vol. x. p. 304) has maintained that the vibrations are *in* the plane of polarization, and deduced it as a consequence of dynamical principles.

But in a subsequent paper, "Sur le Refraction," &c. *Bulletin Math.* July, 1830), the opposite view appears to be implied, and in his notes addressed to M. Libri (*Comptes Rendus*, April 4, 1836), M. Cauchy distinctly maintains that the vibrations are perpendicular to the plane of polarization. And again in the *Nouveaux Exerc.* liv^{n}. 7, his views agree with Fresnel's, in the formula for reflexion.

Professor Maccullagh in his profound Memoir on the laws of crystalline reflexion and refraction (Mem. R. I. Acad. Vol. XVIII.) observes that the last named investigation does not include the case of crystalline reflexion; and in his own elaborate discussion of this subject he shews that vibrations *in* the plane of polarization are absolutely essential to the theory.

37. In the case of elliptical vibrations we have to consider *not*, as in the other cases, a *rectilinear* displacement and its resolved parts, but a *curvilinear* displacement,

which is the result of two virtual rectilinear displacements at right angles to each other, and in a plane perpendicular to the ray, and one of which is *retarded* behind the other by an interval b. Thus the expression will be

$$\eta = \Sigma\left\{a \sin(nt - kx)\right\},$$
$$\xi = \Sigma\left\{\beta \sin(nt - kx - b)\right\}.$$

Taking a single term of each we have,

$$\frac{\eta}{a} = \sin(nt - kx),$$

on substituting which, we have,

$$\zeta = \beta\left(\frac{\eta}{a}\cos b - \sqrt{1 - \frac{\eta^2}{a^2}}\sin b\right).$$

38. Hence we obtain

$$\frac{\zeta^2}{\beta^2 \sin^2 b} - \frac{2\cos b\,\eta\,\zeta}{a\beta \sin^2 b} + \frac{\eta^2}{a^2 \sin^2 b} = 1.$$

The equation to the ellipse described by a molecule, the origin being at the centre, the conjugate axes parallel to the co-ordinate axes of y and z, and their values being,

$$a \sin b = \tfrac{1}{2}\text{ axis}, \qquad \beta \sin b = \tfrac{1}{2}\text{ conjugate axis.}$$

39. If $b = \frac{\pi}{2}$ $\cos b = 0$ $\sin b = 1$, the equation becomes

$$\frac{\zeta^2}{\beta^2} + \frac{\eta^2}{a^2} = 1,$$

the curve is still an ellipse.

40. In this case the formula is sometimes written

$$\eta = \Sigma\left\{a \sin(nt - bx)\right\},$$
$$\zeta = \Sigma\left\{\beta \cos(nt - bx)\right\}.$$

41. If $\alpha = \beta$, we have $\zeta^2 + \eta^2 = \alpha^2$, or it becomes a circle.

42. If $b = 0$, taking a single term, we have from (37)

$$\frac{\eta}{\alpha} = \frac{\zeta}{\beta},$$

or the path of the vibrating molecule is a straight line.

43. The retardation b is constant for the whole ray; and if we suppose that for all the ellipses the values of α and of β respectively are equal, these quantities will become constant coefficients in the summation, and we may write

$$\Sigma\alpha = h\alpha, \qquad \Sigma\beta = h\beta,$$

and the formulas (37) will become

$$\eta = \alpha\,\Sigma\left\{\sin\left(nt - kx\right)\right\},$$

$$\zeta = \beta\,\Sigma\left\{\sin\left(nt - kx - b\right)\right\}.$$

SECTION II.

DYNAMICAL PRINCIPLES.

GENERAL REMARKS.

44. THE first objects in the attempt to refer the vibrations of an ætherial fluid to dynamical principles were mainly the explanation of the phenomena of reflexion and polarization;—of refraction, ordinary and extraordinary; or, in its most general aspect, the wave-surface; the principle of interferences and that of transverse vibrations.

To include these and yet extend the theory to the unequal refrangibility, as well as, if possible, to the absorption and some other points, has been the aim of the more recent investigations.

UNEQUAL REFRANGIBILITY.

45. For the same primary ray, that is, for the same value of λ, the velocity of propagation of waves is (as we have seen) expressed by the reciprocal of the refractive index for the particular medium. In passing from one ray to another, since as in the forms (12) and (13) we have

$$\frac{1}{\mu} = \frac{n}{k} = \frac{n}{2\pi}\lambda,$$

we may observe, that if this fraction $\frac{n}{k}$ be constant, there is no dispersion. In this case the wave-functions are (as already observed) solutions of the partial differential equa-

tion (23) which is the expression for vibratory motion deduced on dynamical grounds, and applicable to the analogous case of the laws of sound. (See Peacock's Examples on *Int. Calc.* p. 475, and Airy's *Tract*, Art. 10). Such a theory might suffice for light regarded as homogeneous.

46. But a theory, to be at all complete, must include the explanation of the unequal refrangibility, and this manifestly requires that it should be such as to give a value of μ as some inverse function of λ.

Now, if in the above expression n were an independent constant, it would give the index in the simple inverse ratio of the wave length which is manifestly not the law of unequal refrangibility, as is evident on the slightest attention to the facts of the dispersion in different media.

47. To obtain, then, some theoretical ground on which such a relation between the wave length and the index could be substantiated, was the object of several highly ingenious suggestions in the earlier stages of the enquiry. And it is even now far from certain that some such causes may not concur in producing the phenomena.

48. But for a more satisfactory and truly legitimate application of theory to this object, we must look to the more recent modifications which the undulatory system has undergone, by the introduction of improved dynamical views of the motions propagated in a system of molecules united in such a way as to constitute an elastic ethereal medium.

49. In this way in fact it has been shewn by the distinguished mathematicians before named, that under certain conditions the ordinary formulas for waves can be

derived from the equations which express the motions of an elastic system on exact dynamical principles, and which also involve the desired relation between the index and the wave length.

50. The existence *in general* of such a relation, would assign a reason why rays, whose waves are of different lengths should be unequally refracted. But for any satisfactory comparison of theory and observation, it is requisite to assign a more specific relation. Theory must shew, not only that *some* relation subsists between the length, and the velocity of a wave, which shall vary for each different ray and each different medium, but also that it shall explain why the several rays are unequally refracted in the *precise degree* which prismatic observations indicate. How far this has been attained it will be our object in the sequel to shew.

DIFFERENTIAL EQUATIONS FOR THE MOTION OF A SYSTEM OF POINTS.

51. We have then, first, to investigate the general equations of motion of a system of molecules placed at distances from each other, which are considerable in comparison with their masses, and acted upon by their mutual attractive and repulsive forces, so as to form an elastic system, liable to slight disturbances.

To take the subject in the most general point of view, let the co-ordinates in space of any molecules m, m', m'', &c. be respectively,

$$\begin{array}{llll} m \ldots\ldots & x, & y, & z, \\ m' \ldots\ldots & x + \Delta x, & y + \Delta y, & z + \Delta z, \\ m'' \ldots\ldots & x + \Delta x', & y + \Delta y', & z + \Delta z', \\ \&c. \ldots\ldots & \&c. & & \end{array}$$

When the system is disturbed, after a time t, let the displacements be respectively, of

$$\begin{array}{llll} m \ldots\ldots & \xi & \eta & \zeta, \\ m' \ldots\ldots & \xi + \Delta\xi & \eta + \Delta\eta & \zeta + \Delta\zeta, \\ m'' \ldots\ldots & \xi + \Delta\xi' & \eta + \Delta\eta' & \zeta + \Delta\zeta'. \end{array}$$

52. Let r be the distance between the two molecules m, m', then manifestly,

$$r = \sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)}.$$

53. Let the force which maintains the system as an elastic medium be any function of this distance, as $f(r)$: then it is easily seen that, supposing the absolute force of each molecule to be represented by $m, m_{,}, m_{,,}$, &c. we have for the forces in the directions of the three axes in equilibrium,

$$\Sigma m f(r) \frac{\Delta x}{r} = 0,$$

$$\Sigma m f(r) \frac{\Delta y}{r} = 0,$$

$$\Sigma m f(r) \frac{\Delta z}{r} = 0.$$

54. Now let the system be disturbed, and after a time t the distance r becomes

$$r + \Delta r = \sqrt{\{(\Delta x + \Delta\xi)^2 + (\Delta y + \Delta\eta)^2 + (\Delta z + \Delta\zeta)^2\}}.$$

55. In this condition we have the three forces,

$$\frac{d^2\xi}{dt^2} = \Sigma\left\{m f(r + \Delta r) \frac{\Delta x + \Delta\xi}{r + \Delta r}\right\},$$

$$\frac{d^2\eta}{dt^2} = \Sigma\left\{m f(r + \Delta r) \frac{\Delta y + \Delta\eta}{r + \Delta r}\right\},$$

$$\frac{d^2\zeta}{dt^2} = \Sigma\left\{m f(r + \Delta r) \frac{\Delta z + \Delta\zeta}{r + \Delta r}\right\}.$$

56. On expanding the value of $r + \Delta r$ and neglecting powers above the first, we find,

$$r\Delta r = \Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta;$$

also by development,

$$\frac{1}{r+\Delta r} = \frac{1}{r} - \frac{\Delta r}{r^2} + \frac{\Delta r^2}{r^3} - \text{\&c.}$$

$$\text{and } f(r+\Delta r) = f(r) + \frac{df(r)}{dr}\Delta r + \text{\&c.}$$

57. Hence we have

$$\frac{f(r+\Delta r)}{r+\Delta r} = \left\{\begin{array}{l} f(r)\left\{\dfrac{1}{r} - \text{\&c.}\right\} \\ + \dfrac{df(r)}{dr}\Delta r\left\{\dfrac{1}{r} - \text{\&c.}\right\}, \end{array}\right.$$

which (neglecting squares) we may write for brevity,

$$\phi(r) = \frac{f(r)}{r} + \psi(r)\,r\Delta r.$$

58. On substituting in the equations (55) and observing that the terms (53) are involved, which disappear, these equations become,

$$\frac{d^2\xi}{dt^2} = \Sigma\left[m\left\{\psi(r)\,r\Delta r\,\Delta x + \phi(r)\,\Delta\xi\right\}\right],$$

$$\frac{d^2\eta}{dt^2} = \Sigma\left[m\left\{\psi(r)\,r\Delta r\,\Delta y + \phi(r)\,\Delta\eta\right\}\right],$$

$$\frac{d^2\zeta}{dt^2} = \Sigma\left[m\left\{\psi(r)\,r\Delta r\,\Delta z + \phi(r)\,\Delta\zeta\right\}\right].$$

59. Here substituting again the value of $r\Delta r$ (56) these equations ultimately become,

$$\frac{d^2\xi}{dt^2} = \Sigma\left\{m\left\{\begin{array}{l}\phi(r)\,\Delta\xi \\ + \psi(r)\,\Delta x\,(\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta),\end{array}\right.\right.$$

$$\frac{d^2\eta}{dt^2} = \Sigma \left\{ m \left\{ \begin{array}{l} \phi(r)\,\Delta\eta \\ + \psi(r)\,\Delta y\,(\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta), \end{array} \right. \right.$$

$$\frac{d^2\zeta}{dt^2} = \Sigma \left\{ m \left\{ \begin{array}{l} \phi(r)\,\Delta\zeta \\ + \psi(r)\,\Delta z\,(\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta). \end{array} \right. \right.$$

Such are the general and fundamental equations deduced, though under a slightly different form, by M. Cauchy, and in this identical form by several other mathematicians:—the first adoption of the principle and the deduction of equations nearly resembling these, under a more restricted view, having been undoubtedly due to M. Navier.

TRANSVERSE VIBRATIONS.

60. It is on the principle of a system of molecules constituted as above supposed, and through the simple effect of forces acting in the manner just developed that we establish the existence of the transverse vibrations. Of this it may be desirable to offer a rather more particular illustration.

61. Conceiving then such a system as above, when any row or line of molecules is similarly displaced, and through a space which is small compared with the separating intervals, the molecules of the preceding row will be moved in the same direction by the forces which are thus developed with the change of distance; so that the vibrations of the particles composing the first row will be communicated to those of the second, and thus the vibratory motion will be propagated in a direction perpendicular to that in which it takes place.

To account for the fact that there are no sensible vibrations in a direction normal to the plane of the wave, that is, in the direction of the ray, we have only to suppose

the repulsive force between the molecules to be very great, or the resistance to compression very considerable; for in this case, it will be seen, the force which resists the approach of two strata of the fluid is much greater than that which opposes their sliding one on another.

62. It ought however to be mentioned that some difference of opinion has been manifested among mathematicians as to the precise "modus operandi" by which this is effected.

The view above taken was first proposed by Fresnel, in a short paper entitled "Considerations Mecaniques sur la Polarization," &c. (*Bulletin, Math.* 1824) as well as in his memoir on double refraction.

Mr Airy in his tract (Art. 103) has also given a demonstration on the more restricted hypothesis of the force being inversely as the square of the distance.

M. Cauchy shews it to be a consequence of his principles in his memoir on the theory of light (*Mem. Instit.* Vol. x.), Mr Tovey (*Journal of Science*, No. 71. See also Vol. IX. 421) objects to the view adopted by Mr Kelland, (*Ibid*, Vol. IX. p. 341) as appearing to imply a condition of unstable equilibrium. His own view accords with Fresnel's, and is briefly that, if the ray coincide with x, neither the displacements $\eta\zeta$ nor their differences $\Delta\eta$, $\Delta\zeta$ cause any change of density in the medium: but the differences $\Delta\xi$ would imply a change of density. If then we suppose the force by which the ether resists compression to be so great that in the motions producing light it may be regarded as incompressible, the differences $\Delta\xi$ vanish. On the whole subject, especially the further arguments of M. Poisson and others, the student should consult Prof. Lloyd's Report, p. 355, &c.

INTEGRATION OF THE EQUATIONS OF MOTION.

63. In the integration of these equations several methods have been adopted, these of course all turn upon finding some expressions for $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ by development, or otherwise, from which are derived forms susceptible of direct integration.

We shall here adopt, with a view to our particular object, a method which, though apparently prolix and capable of simplification, it is yet desirable to follow as it includes other important consequences.

64. In the first place, without loss of generality, we may adopt the supposition that the ray coincides with the axis x, whence, agreeably to what was before remarked we have,

$$\xi = 0, \ \Delta\xi = 0, \ \frac{d^2\xi}{dt^2} = 0,$$

or the equations are reduced to

$$\frac{d^2\eta}{dt^2} = \Sigma[m\{\phi(r)\Delta\eta + \psi(r)\Delta y(\Delta y\Delta\eta + \Delta z\Delta\zeta)\}],$$

$$\frac{d^2\zeta}{dt^2} = \Sigma[m\{\phi(r)\Delta\zeta + \psi(r)\Delta z(\Delta y\Delta\eta + \Delta z\Delta\zeta)\}].$$

65. In proceeding to the integration, on this supposition, if we consider the two component displacements η, ζ as related in the way expressed by the formula (37) viz.;

$$\eta = \Sigma\{a\sin(nt - kx)\},$$
$$\zeta = \Sigma\{\beta\sin(nt - kx - b)\},$$

(in which, if a, β and b are assumed as before explained, we may regard this as the general form, from which by intro-

ducing the several modifications before explained, we can also express all the different cases, whether of elliptic or circularly polarized, plane-polarized, or unpolarized light. Now it is easy to shew that this formula is the solution of the differential equations in the form just given (64), if n have a certain value, which we proceed to determine.

66. Taking the increments of these expressions, we have (18)

$$\Delta\eta = \Sigma \left\{ a \begin{cases} -2\sin^2 \dfrac{k\Delta x}{2} \sin(nt - kx) \\ -\sin k\Delta x \cos(nt - kx), \end{cases} \right.$$

$$\Delta\zeta = \Sigma \begin{cases} \beta \cos b \begin{cases} -2\sin^2 \dfrac{k\Delta x}{2} \sin(nt - kx) \\ -\sin k\Delta x (\cos nt - kx) \end{cases} \\ +\beta \sin b \begin{cases} \sin k\Delta x \sin(nt - kx) \\ -2\sin^2 \dfrac{k\Delta x}{2} \cos(nt - kx). \end{cases} \end{cases}$$

67. Also differentiating them, we find

$$\frac{d^2\eta}{dt^2} = -n^2 \Sigma a \sin(nt - kx),$$

$$\frac{d^2\zeta}{dt^2} = -n^2 \Sigma \{\beta \cos b \sin(nt - kx) - \beta \sin b \cos(nt - kx)\}.$$

68. Now for brevity writing,

$$p = m\{\phi(r) + \psi(r)\Delta y^2\},$$
$$p' = m\{\phi(r) + \psi(r)\Delta z^2\},$$
$$q = m\{\psi(r)\Delta y \Delta z\},$$
$$2\theta = k\Delta x.$$

69. The equations (64) will be expressed by

$$\frac{d^2\eta}{dt^2} = \{\Sigma(p\Delta\eta) + \Sigma(q\Delta\zeta)\},$$

$$\frac{d^2\zeta}{dt^2} = \{\Sigma(p'\Delta\zeta) + \Sigma(q\Delta\eta)\}.$$

70. And here, substituting the above values of $\Delta\eta\,\Delta\zeta$, and arranging the terms, these equations become

$$\frac{d^2\eta}{dt^2} = \left\{\begin{array}{l} \left.\begin{array}{l} +\sin b\,\Sigma\,(\beta q \quad \sin 2\,\theta) \\ -\cos b\,\Sigma\,(\beta q\, 2\sin^2.\,\theta) \\ - \qquad \Sigma\,(a p\, 2\sin^2.\,\theta) \end{array}\right\} \sin\,(nt - kx), \\ \left.\begin{array}{l} - \qquad \Sigma\,(a p \quad \sin 2\,\theta) \\ -\cos b\,\Sigma\,(\beta q \quad \sin 2\,\theta) \\ -\sin b\,\Sigma\,(\beta q\, 2\sin^2\,\theta) \end{array}\right\} \cos\,(nt - kx), \end{array}\right.$$

$$\frac{d^2\zeta}{dt^2} = \left\{\begin{array}{l} \left.\begin{array}{l} +\sin b\,\Sigma\,(\beta p' \quad \sin 2\,\theta) \\ -\cos b\,\Sigma\,(\beta p' \quad 2\sin^2\theta) \\ - \qquad \Sigma\,(a q \quad 2\sin^2\theta) \end{array}\right\} \sin\,(nt - kx), \\ \left.\begin{array}{l} - \qquad \Sigma\,(a q \quad \sin 2\,\theta) \\ -\cos b\,\Sigma\,(\beta p' \quad \sin 2\,\theta) \\ -\sin b\,\Sigma\,(\beta p' \quad 2\sin^2\theta) \end{array}\right\} \cos\,(nt - kx). \end{array}\right.$$

71. On comparing these expressions with those for the same functions (67), which must be identical, and equating the respective coefficients of $\sin\,(nt - kx)$ and of $\cos\,(nt - kx)$, since they must hold good for all values of those terms, we have the following equations:

72.
$$-n^2\,\Sigma\,a = \left\{\begin{array}{l} +\sin b\,\Sigma\,(\beta q \quad \sin 2\,\theta) \\ -\cos b\,\Sigma\,(\beta q\, 2\sin^2\theta) \\ - \qquad \Sigma\,(a\,p\, 2\sin^2\theta). \end{array}\right.$$

73.
$$0 = \left\{\begin{array}{l} - \qquad \Sigma\,(a p \quad \sin 2\,\theta) \\ -\cos b\,\Sigma\,(\beta q \quad \sin 2\,\theta) \\ -\sin b\,\Sigma\,(\beta q \quad 2\sin^2\theta). \end{array}\right.$$

74.

$$-n^2 \cos b\, \Sigma\, \beta = \begin{cases} + \sin b\, \Sigma\, (\beta p' \sin 2\,\theta) \\ - \cos b\, \Sigma\, (\beta p'\ 2 \sin^2 \theta) \\ - \qquad \Sigma\, (aq\ 2 \sin^2 \theta). \end{cases}$$

75.

$$-n^2 \sin b \Sigma \beta = \begin{cases} - \qquad \Sigma\, (aq\ \ \sin 2\theta) \\ - \cos\, b\, \Sigma\, (\beta p' \sin 2\theta) \\ - \sin\, b\, \Sigma\, (\beta p'\, 2 \sin^2 \theta). \end{cases}$$

76. From the two last forms (74) (75), by multiplication and addition, we obtain

$$-n^2\, \Sigma\, \beta = \begin{cases} - \qquad \Sigma\, (\beta p'\ 2 \sin^2 \theta) \\ - \cos b\, \Sigma\, (aq\ \ 2 \sin^2 \theta) \\ - \sin\, b\, \Sigma\, (aq \qquad \sin 2\theta). \end{cases}$$

77. From these formulas, without going into detail, it is manifest we might deduce a value of n as a certain function of m, a, β, p, p', q and θ: let us call such a value n; then it follows on substituting it in formula (65) that the expressions

$$\eta = \Sigma\, \{a \sin\, (\mathrm{n}t - kx)\},$$
$$\zeta = \Sigma\, \{\beta \sin\, (\mathrm{n}t - kx - b)\},$$

are the solutions of the differential equations (64) which represent the motions of an elastic system, and the problem is solved, as far as the integration is concerned.

SECTION III.

DEDUCTIONS FROM THE INTEGRATION.

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78. In the preceding solution of the equations of motion, that is, the deduction of the wave-function from our original dynamical principles, it is evident that as the value n involves θ, that is, $\frac{\pi \Delta x}{\lambda}$, we have included in the coefficient thus introduced into the wave-function, a relation between the wave-length and the velocity on refractive index, applicable to the case of the dispersion.

79. In the present Section it will be our object to develop more particularly both this result and some others of considerable interest. And with this view we must recur to the formulas last obtained, and trace the consequences in the several cases arising from the different conditions of light in regard to polarization.

CASES OF POLARIZED AND UNPOLARIZED LIGHT.

80. In the case of elliptic polarization, from the conditions before stated (43), we can obtain from the forms (72) and (76),

$$n^2 = \frac{1}{h\,(\alpha^2 + \beta^2)} \left\{ \begin{array}{ll} \beta^2 & \Sigma\,(p'\,2\sin^2\theta) \\ +\,\alpha^2 & \Sigma\,(p\;\;2\sin^2\theta) \\ +\,2\cos b\,\alpha\beta\,\Sigma & (q\;\;2\sin^2\theta). \end{array} \right.$$

81. Also the form (75) gives, on transposing,

$$\sin b = \frac{\{\alpha\Sigma(q \sin 2\theta) + \beta \cos b \Sigma(p' \sin 2\theta)\}}{n^2 h\beta - \beta\Sigma(p' 2 \sin^2\theta)}.$$

Upon the whole, then, we see that the formula (43) for elliptically polarized light, involving the above value of n, is the solution of the differential equations (64) for the motion of a system of molecules constituted as at first supposed.

82. In the case of unpolarized light—if instead of the formula (65) we had taken the expressions for the component displacements $\eta\zeta$, as not involving such a relation, but simply as in the original expressions (34), or supposing $b = 0$ in the forms (65), we might still pursue steps analogous to those above exhibited, though with different values. To trace these results we have only to alter these formulas agreeably to the new conditions. Thus, for unpolarized light, on making $\sin b = 0$, $\cos b = 1$, we have

83. From (72)

$$0 = \begin{cases} n^2 \Sigma\alpha \\ - \quad \Sigma(\alpha p\, 2 \sin^2\theta) \\ - \quad \Sigma(\beta q\, 2 \sin^2\theta). \end{cases}$$

84. From (73)

$$0 = \begin{cases} - \quad \Sigma(\alpha p \sin 2\theta) \\ - \quad \Sigma(\beta q \sin 2\theta). \end{cases}$$

85. From (75)

$$0 = \begin{cases} - \quad \Sigma(\beta p' \sin 2\theta) \\ - \quad \Sigma(\alpha q \quad \sin 2\theta). \end{cases}$$

86. From (74)

$$0 = \begin{cases} \quad n^2 \Sigma\beta \\ - \quad \Sigma(\beta p'\, 2 \sin^2\theta) \\ - \quad \Sigma(\alpha q\, 2 \sin^2\theta). \end{cases}$$

Now in all these equations it is evident that since α and β are by the original condition ***wholly arbitrary and independent*** both of each other and of the other quantities, these equations can only hold good for ***all values whatever***, of α and β, if each of the terms involving respectively α and β are ***separately*** $= 0$, that is, we must have

87. From (84) $0 = \Sigma\,(\alpha p \sin 2\theta)$.

88. $0 = \Sigma\,(\beta q \sin 2\theta)$.

89. From (85) $0 = \Sigma\,(\beta p' \sin 2\theta)$.

90. $0 = \Sigma\,(\alpha q \sin 2\theta)$.

91. From (83) $0 = \Sigma\,(\beta q\, 2 \sin^2\theta)$.

92. From (86) $0 = \Sigma\,(\alpha q\, 2 \sin^2\theta)$.

93. From (83) $0 = n^2\, \Sigma\alpha - \Sigma\,(\alpha p\, 2 \sin^2\theta)$.

94. From. (86) $0 = n^2\, \Sigma\beta - \Sigma\,(\beta p'\, 2 \sin^2\theta)$.

95. From the last two forms we have

$$n^2 = \frac{1}{\Sigma\alpha}\, \Sigma\,(\alpha p\, 2 \sin^2\theta).$$

96.
$$n^2 = \frac{1}{\Sigma\beta}\, \Sigma\,(\beta p'\, 2 \sin^2\theta).$$

97. These two last values being identical, calling the value of n, in this case, n′, and substituting as above, the equations (34) involving n′ are the solutions for the case of unpolarized light.

98. For plane polarized light from the conditions (35) let $\beta = 0$, and the form (73) will become

$$0 = \Sigma\,(\alpha p \sin 2\theta).$$

99. In like manner (74) will give

$$0 = \Sigma\,(aq\;2\sin^2\theta).$$

100. And (75) $0 = \Sigma\,(aq\sin 2\theta)$.

101. While from (72) we find

$$n^2 = \frac{1}{\Sigma a}\,\Sigma\,(ap\;2\sin^2\theta).$$

With this value of n, which is the same as in the last case, n', substituted as before in the equations (34) the second equation disappearing, the first is the solution for plane polarized light.

102. But here another consideration arises; the introduction of the values deduced in forms (83, &c.) for unpolarized light, and in (98, &c.) for polarized, will considerably modify the form of the original differential equations.

103. Since $\Delta\eta$ and $\Delta\zeta$ are both of the form

$$M\sin 2\theta + N\,2\sin^2\theta, \quad (20)$$

the above conditions give

$$\Sigma\,(q\Delta\eta) = 0,$$

$$\Sigma\,(q\Delta\zeta) = 0.$$

104. Thus, when they are fulfilled in the terms of the original equations, those equations become for unpolarized light,

$$\frac{d^2\eta}{dt^2} = \Sigma\,(p\Delta\eta),$$

$$\frac{d^2\zeta}{dt^2} = \Sigma\,(p'\Delta\zeta),$$

while for plane polarized light the second of these forms disappears.

105. Hence it follows, that the formulas for unpolarized light, as well as the formula for plane polarized light are *only* solutions, provided these conditions are fulfilled in the original equations, that is, provided they are in the forms (104).

106. While, on the other hand, in the case of elliptically polarized light, it is important to bear in mind that the solution has been obtained solely from the conditions of elliptic polarization, the original equation *being in the forms* (64) or (69) *retaining all its terms.*

HYPOTHESES AS TO THE ARRANGEMENT OF THE MOLECULES.

107. The differential equations were originally obtained for the motions of a system of molecules on a very general hypothesis of the forces which united them so as to form an elastic medium. And it is material to observe, that these equations have been obtained *without any particular supposition being made as to the arrangement of the molecules in space: they consequently apply if we imagine the molecules distributed in the most irregular or unsymmetrical manner.*

108. Now we have in the above deductions found that certain terms vanish, which involve quantities dependent on our hypothesis as to the supposed constitution of the medium. To examine then the conditions under which these terms can vanish, we may first observe, that since none of the factors can separately become $= 0$, the terms can only become nothing by sums with opposite signs being equal and destroying each other.

That this may happen, depends on an hypothesis respecting the arrangement of the *æthereal molecules* in

spaces, viz. that they are ***distributed uniformly***. This is the supposition adopted by M. Cauchy and other writers.

The slightest consideration of the nature of such an arrangement will suffice to shew, that on this hypothesis the axis x passing through the first molecule m in any direction, the sums of the corresponding distances of all the other molecules on each side of it, whether in the plane of y or z, will be equal for all positions of x in the medium.

109. Hence it is easily seen that the respective sums of products

$$\phi r \sin 2\theta, \quad \psi r \Delta y^2 \sin 2\theta, \quad \psi r \Delta z^2 \sin 2\theta,$$

$$\psi r \Delta y \Delta z \sin 2\theta,$$

with opposite signs will be equal. Thus we shall always have

$$\Sigma (aq \sin 2\theta) = 0,$$

$$\Sigma (ap \sin 2\theta) = 0,$$

$$\Sigma (ap' \sin 2\theta) = 0.$$

110. But whenever these terms are evanescent, it is easy to shew that we always have also

$$\Sigma (aq \; 2 \sin^2\theta) = 0;$$

and similarly for the like terms involving β, by a simple transformation of co-ordinates, as explained by Sir J. Lubbock in his valuable paper*. That paper indeed relates to the more general views of the subject, to which I shall refer in the sequel; but the particular process in question is independent of these views, and is as follows:

* Journal of Science, Nov. 1839.

111. Let x', y', z' be the new co-ordinates, and t an assumed arc, such that,

$$y = y' \cos t - z' \sin t,$$

$$y' = y \cos t + z \sin t,$$

$$z = z' \cos t + y' \sin t,$$

$$z' = z \cos t - y \sin t.$$

Whence $\Delta y' = \Delta y \cos t + \Delta z \sin t,$

$$\Delta z' = \Delta z \cos t - \Delta y \sin t.$$

112. If then for abridgment writing,

$$a\psi(r)\, 2 \sin^2\theta = P,$$

the arc t be so assumed that we have

$$\tan 2t = \frac{2\Sigma(P\Delta z'\Delta y')}{\Sigma(P\Delta y'^2) - \Sigma(P\Delta z'^2)},$$

it will be easily found that we must have

$$\Sigma(P\Delta y\,\Delta z) = 0.$$

113. Or in other words,

$$\Sigma(2aq \sin^2\theta) = 0.$$

114. Thus the hypothesis of symmetrical distribution gives

$$\Sigma(q\Delta\eta) = 0, \quad \Sigma(q\Delta\zeta) = 0.$$

115. And the original equations are reduced to

$$\frac{d^2\eta}{dt^2} = \Sigma(p\Delta\eta),$$

$$\frac{d^2\zeta}{dt^2} = \Sigma(p'\Delta\zeta).$$

The same as (104).

116. Or restoring the original values,

$$\frac{d^2\eta}{dt^2} = \Sigma\left[m\left\{\phi(r) + \psi(r)\,\Delta y^2\right\}\Delta\eta\right],$$

$$\frac{d^2\zeta}{dt^2} = \Sigma\left[m\left\{\phi(r) + \psi(r)\,\Delta x^2\right\}\Delta\zeta\right].$$

That is, the equations are reduced to precisely the same form by the hypothesis of uniform distribution, as they are on the hypothesis of unsymmetrical distribution, by the conditions of plane polarized and unpolarized light.

Thus the formulas for *plane polarized and unpolarized* light are *only* solutions of the original equations when in the same form, to which they are reduced by *symmetrical distribution*.

117. For elliptically polarized light, on the hypothesis of *symmetrical* distribution, we can follow out results analogous to those above obtained. We should have, instead of the form (72),

$$n^2 h = \Sigma\,(p\;2\sin^2\theta).$$

118. And instead of (76),

$$n^2 h = \Sigma\,(p'\;2\sin^2\theta),$$

which is identical with the former, whence we have $p = p'$ also.

119. $$n^2 = \frac{1}{h}\,\Sigma\,(p\;2\sin^2\theta).$$

120. The formula (81) is thus reduced to

$$\sin b = \frac{0}{0}.$$

Thus (although with altered values) the formula for *elliptic polarization* is a solution of the original equation *equally* in the form (64), and when reduced to the form (116) by the hypothesis of *symmetrical distribution*. In other words, of the equations, in the form (64), the formula for elliptic polarization is the only solution: in the form (116) the formulas for elliptic vibrations or rectilinear indifferently are solutions.

121. Thus it follows, that if we suppose the ætherial molecules *unsymmetrically* distributed, then *elliptic polarization alone* is the result. *Æther so constituted cannot admit rectilinear vibrations.* Light, therefore, entering such a portion of æther necessarily becomes elliptically polarized.

If we suppose the molecules symmetrically distributed, this is compatible with either elliptic or rectilinear vibrations indifferently. Either therefore will be propagated according to the condition of the intromitted ray.

Thus elliptic polarization is traced to its cause in the simple consideration, that the vibrations which constitute it are necessarily produced when waves are propagated through any portion of æther in which a symmetrical arrangement of the molecules does not subsist.

122. The investigation conducted by Mr Tovey's method*, is directed to shewing by the equations (4) of his paper, that when the sums involving the odd powers of the differences are *not* evanescent, the quantities b and ρ (the ratio of the semiaxes of the ellipse) are *determinate;* or, in other words, the expressions must belong to ellipses, or in a medium so constituted as to make those sums finite, elliptical polarization will result. When the sums just

* *Journal of Science*, Vol. XII. p. 10.

mentioned vanish, then it is seen that those quantities are altogether arbitrary, and the movements will be the same as those expressed by the author's formulas in his other paper*, or such a medium will propagate elliptic or rectilinear vibrations indifferently.

123. Mr Whewell† had observed the difficulty of conceiving any mechanical conditions for the production of elliptic polarization, and that not even a plausible hypothesis had been proposed so as to give a physical interpretation to the language of analysis, especially as conveyed in the equations obtained by Prof. Maccullagh.

From what has just been stated, this difficulty appears now to be, at least in a general way, overcome by the conclusion of Mr Tovey.

124. It is easy to imagine the physical possibility of a portion of the æther possessing an unsymmetrical arrangement of its molecules. For example, at the bounding surface of a medium and of vacuum, or generally of two media of different densities, whatever view we adopt as to the density of the æther we must suppose some kind of modification which we can hardly suppose to take place abruptly; but may fairly imagine a thin stratum on either side, within which there is a gradual alteration in the arrangement of the molecules; and this more considerable as the difference of the refractive powers is greater. It is conceivable that this variation may in some instances be of sufficiently great amount to give the requisite conditions of unsymmetrical distribution within this stratum, though on either side of it the symmetrical

* Lond. and Edinb. Phil. Mag. and Journal of Science, Vol. VIII. p. 426 and p. 502. See also the same author's paper on the Absorption of Light. *Journal of Science.*

† *Hist. of Ind. Science*, Vol. II. p. 448, 1837.

arrangement may subsist. This however is merely thrown out as an illustration.

FORMULA FOR DISPERSION.

125. To return to our original enquiry respecting the unequal refrangibility: with regard to the value of n, experiment shews that the state of the same ray as to polarization produces no difference in the magnitude of its refractive index. Hence it follows that in all the preceding different cases in which the value of n has been expressed (whether on the hypothesis of unsymmetrical or of symmetrical distribution) the terms involved must vary in magnitude, so that the whole expression shall remain constant, or the values n, n′, &c. all equal. Thus, in general, writing h for the sum of the arbitrary terms α and β, we express the value of n by the formula

$$n^2 = \frac{1}{h}\Sigma\,(a p\ 2\sin^2\theta).$$

126. Again, recurring to the values (13) (68)

$$h = \frac{2\pi}{\lambda},\quad \theta = \frac{\pi\Delta x}{\lambda},$$

we have obviously

$$\frac{1}{\mu^2} = \frac{n^2}{\left(\frac{2\pi}{\lambda}\right)^2} = \frac{1}{4}\,\frac{n^2\Delta x^2}{\left(\frac{\pi\Delta x}{\lambda}\right)^2} = \frac{1}{4}\,\frac{n^2\Delta x^2}{\theta^2}.$$

127. And on substituting in the form (125) we obtain

$$\frac{1}{\mu^2} = \frac{1}{2h}\,\Sigma\left\{am\,\{\phi(r) + \psi(r)\Delta y^2\}\,\Delta x^2\left(\frac{\sin^2\left(\frac{\pi\Delta x}{\lambda}\right)}{\left(\frac{\pi\Delta x}{\lambda}\right)^2}\right)\right\}.$$

128. This for abridgment may be expressed by

$$\frac{1}{\mu^2} = \Sigma \left\{ H^2 \frac{\sin^2\left(\frac{\pi \Delta x}{\lambda}\right)}{\left(\frac{\pi \Delta x}{\lambda}\right)^2} \right\},$$

which is the formula for the dispersion.

The more particular examination of this formula in all its bearings will be the subject of a future section. Meanwhile one or two other topics claim our notice.

SECTION IV.

OTHER SOLUTIONS OF THE DIFFERENTIAL EQUATIONS.

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129. RETURNING to the general supposition of the displacements in three rectangular axes, by other methods than that before followed, the direct integration of the general equations has been effected by M. Cauchy and several other mathematicians who have improved upon his process. But all these investigations expressly proceed upon the adoption of the *symmetrical arrangement* as the fundamental hypothesis.

Now it has been already shewn that on this supposition we have

$$\Sigma\{(\psi r\, \Delta y\, \Delta x)\Delta\eta\} = 0,$$

and all terms of the same form in like manner vanish; also on this hypothesis we may suppose the absolute forces of all the molecules equal, or make the quantity m a constant coefficient. Thus the equations (59) become

$$\frac{d^2\xi}{dt^2} = m\,\Sigma\,[\{\phi\,(r) + \psi\,(r)\,\Delta x^2\}\,\Delta\xi],$$

$$\frac{d^2\eta}{dt^2} = m\,\Sigma[\{\phi\,(r) + \psi\,(r)\,\Delta y^2\}\,\Delta\eta],$$

$$\frac{d^2\zeta}{dt^2} = m\,\Sigma\,[\{\phi\,(r) + \psi\,(r)\,\Delta z^2\}\,\Delta\zeta].$$

130. Mr Kelland deduces as above the fundamental equations (59).

Then developing $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ each as functions of the three variables x, y, z, we have

$$\Delta\xi = \begin{cases} \dfrac{d\xi}{dx}\Delta x + \dfrac{d\xi}{dy}\Delta y + \dfrac{d\xi}{dz}\Delta z \\ + \dfrac{d^2\xi}{dx^2}\dfrac{\Delta x^2}{2} + \dfrac{d^2\xi}{dy^2}\dfrac{\Delta y^2}{2} + \dfrac{d^2\xi}{dz^2}\dfrac{\Delta z^2}{2} \\ + \dfrac{d^2\xi}{dx\,dy}\Delta x\,\Delta y + \dfrac{d^2\xi}{dx\,dz}\Delta x\,\Delta z + \dfrac{d^2\xi}{dy\,dz}\Delta y\,\Delta z \\ + \text{\&c.} \end{cases}$$

and similar expressions for $\Delta\eta$ and $\Delta\zeta$.

131. Then adopting the hypothesis of symmetrical distribution and thus rejecting the sums of products of odd dimensions in $\Delta x\,\Delta y\,\Delta z$;—rejecting also products of the same terms of four dimensions as insensibly small, and lastly from the symmetrical distribution having also

$$\Sigma\phi r\,\Delta x^2 = \Sigma\phi r\,\Delta y^2 = \Sigma\phi r\,\Delta z^2 = 2n^2.$$

The equations are at length reduced to the form

$$\frac{d^2\xi}{dt^2} = n^2\left\{\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} + \frac{d^2\xi}{dz^2}\right\},$$

$$\frac{d^2\eta}{dt^2} = n^2\left\{\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2} + \frac{d^2\eta}{dz^2}\right\},$$

$$\frac{d^2\zeta}{dt^2} = n^2\left\{\frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} + \frac{d^2\zeta}{dz^2}\right\}.$$

Of these expressions solutions are immediately found in the wave functions as at first observed (26).

132. In proceeding to exhibit the value of n by taking the increments $\Delta\xi$, &c. as above (66), the author deduces directly from the symmetrical distribution not only such terms as

$$\Sigma\,(q \sin 2\theta) = 0,$$

but also (p. 162, line 9) the other terms

$$\Sigma\,(q 2 \sin^2\theta) = 0,$$

without any transformation to other axes, by considerations deduced directly from the symmetrical arrangement.

133. Mr Tovey (in his *first* papers in the *Journal of Science*, Vol. VIII. p. 7, 270, 500). After establishing the fundamental equations (59) proceeds to develop ξ, η, ζ as each a function of x. He thus finds

$$\Delta\xi = \frac{d\xi}{dx}\Delta x + \frac{d^2\xi}{dx^2}\frac{\Delta x^2}{2} + \frac{d^3\xi}{dx^3}\frac{\Delta x^3}{2\,.\,3} + \text{\&c.}$$

and similar expressions for $\Delta\eta$ and $\Delta\zeta$.

134. On substituting, he rejects the sums of products of odd dimensions on the hypothesis of symmetrical arrangement: and hence writing for abridgment

$$\Sigma\left\{\phi(r) + \psi(r)\,\Delta x^2\right\}\frac{\Delta x^2}{2} = h_1,$$

$$\Sigma\left\{\phi(r) + \psi(r)\,\Delta x^2\right\}\frac{\Delta x^4}{2\,.\,3\,.\,4} = h_2,$$

$$\text{\&c.} \qquad\qquad = \text{\&c.}$$

The equations become

$$\frac{d^2\xi}{dt^2} = h_1\frac{d^2\xi}{dx^2} + h_2\frac{d^4\xi}{dx^4} + \text{\&c.}$$

$$\text{\&c.} = \text{\&c.},$$

Which are the same forms as those whose integration was explained before, (28). They give the wave-function involving at once the value of n in a series of even powers of k.

135. Sir J. Lubbock (*Journal of Science*, Vol. XI. Nov. 1837), gives the following solution.

Developing $\Delta\rho$ as a function of r,

$$\Delta\rho = \frac{d\rho}{dr}\Delta r + \frac{d^2\rho}{dr^2}\,\frac{\Delta r^2}{1\,.\,2} + \text{\&c.}$$

Then since the symmetrical arrangement causes to disappear the terms involving odd products,

$$m\phi\,(r)\,\Delta r, \quad m\phi\,(r)\,\Delta x^2\Delta r, \text{ \&c.}$$

On substituting for $\Delta\xi$, &c. the values (31) in the equations (129) the author obtains the forms,

$$\frac{d^2\xi}{dt^2} = a^2\cos X\frac{d^2\rho}{dr^2} + \mathrm{a}^2\cos X\frac{d^4\rho}{dr^4},$$

$$\frac{d^2\eta}{dt^2} = b^2\cos Y\frac{d^2\rho}{dr^2} + \mathrm{b}^2\cos X\frac{d^4\rho}{dr^4},$$

$$\frac{d^2\zeta}{dt^2} = c^2\cos Z\frac{d^2\rho}{dr^2} + \mathrm{c}^2\cos X\frac{d^4\rho}{dr^4}.$$

In which

$$a^2 = \frac{m}{1\,.\,2}\,\Sigma\,\{\phi\,(r) + \psi\,(r)\,\Delta x^2\}\,\Delta r^2,$$

$$\mathrm{a}^2 = \frac{m}{1\,.\,2\,.\,3\,.\,4}\,\Sigma\,\{\phi\,(r) + \psi\,(r)\,\Delta x^2\}\,\Delta r^4,$$

and similar values of b, b for Δy, and of c, c for Δz. a^2, b^2, c^2 are the same as in Fresnel's investigation.

136. From the value of ρ (32) there also results

$$\frac{d^2\rho}{dt^2} = \cos X\frac{d^2\xi}{dt^2} + \cos Y\frac{d^2\eta}{dt^2} + \cos Z\frac{d^2\zeta}{dt^2}.$$

From this and the forms above (135) we obtain

$$\frac{d^2\rho}{dt^2} = \begin{cases} (a^2\cos^2 X + b^2\cos^2 Y + c^2\cos^2 Z)\dfrac{d^2\rho}{dr^2} \\ + (\mathrm{a}^2\cos^2 X + \mathrm{b}^2\cos^2 Y + \mathrm{c}^2\cos^2 Z)\dfrac{d^4\rho}{dr^4}. \end{cases}$$

Or writing

$$v^2 = a^2\cos^2 X + b^2\cos^2 Y + c^2\cos^2 Z$$

$$\mathrm{v}^2 = \mathrm{a}^2\cos^2 X + \mathrm{b}^2\cos^2 Y + \mathrm{c}^2\cos^2 Z.$$

It may be expressed in the form,

$$\frac{d^2\rho}{dt^2} = v^2\frac{d^2\rho}{dr^2} + \mathrm{v}^2\frac{d^4\rho}{dr^4},$$

which is of the form at first integrated (28).

The same author has given an investigation somewhat similar by taking the increment $\Delta\rho$ in the same manner as in the foregoing section. (*Journal of Science*, XII. Jan. 1838).

137. In the investigation just given, the author's object was to compare Fresnel's views with those since proposed; and the difference is here clearly exhibited by the two members of each equation distinguished by the coefficients a^2 and a^2, &c. If the second set be neglected the expressions would precisely agree with the views of Fresnel, or we should have,

$$\frac{d^2\xi}{dt^2} = a^2\cos X\frac{d^2\rho}{dr^2}.$$

&c. = &c.

138. These expressions are the components of the force in the direction of the axes; and the resultant will be

$$\sqrt{\left\{a^4 \cos^2 X \left(\frac{d^2\rho}{dr^2}\right)^2 + b^4 \cos^2 Y \left(\frac{d^2\rho}{dr^2}\right)^2 + c^4 \cos^2 Z \left(\frac{d^2\rho}{dr^2}\right)^2\right\}}.$$

And since generally the component divided by the resultant gives the cosine of the inclination of the resultant or line of force on that axis, we should have

$$\frac{-a^2 \cos X}{\sqrt{a^4 \cos^2 X + b^4 \cos^2 Y + c^4 \cos^2 Z}};$$

which may be written $\dfrac{-a^2 \cos X}{f}$;

and similarly $\dfrac{-b^2 \cos Y}{f}$;

and $\dfrac{-c^2 \cos Z}{f}$;

for the cosines of inclination of the resultant on the three axes respectively.

139. But the reasoning manifestly does not turn upon the particular value here assigned to a^2, &c. but would apply equally if we took the complete expressions: thus writing

$$m\,\Sigma\left\{\phi(r) + \psi(r)\,\Delta x^2\right\} = a_{\prime}^{2},$$
$$\&c. \quad = \&c.$$

we shall have

$$\frac{d^2\xi}{dt^2} = a_{\prime}^{2} \cos X\,\Delta\rho,$$
$$\&c.$$

for the several components: and for the resultant,

$$\sqrt{\left\{a_{\prime}^{4} \cos^2 X\,(\Delta\rho)^2 + b^4 \cos^2 Y\,(\Delta\rho)^2 + c^4 \cos^2 Z\,(\Delta\rho^2\right\}},$$

and thus the cosines of inclination will be

$$\frac{-a_{,}^{2}\cos X}{f_{,}},\quad \frac{-b_{,}^{2}\cos Y}{f_{,}},\quad \frac{-c_{,}^{2}\cos Z}{f_{,}}.$$

The application of this will appear in the next Section.

140. The original investigation of M. Cauchy which in fact led the way to the whole theory of dispersion, though closely connected with various others in the *Exercices de Mathematics*, is yet given completely in all that relates to our immediate subject, in his *Mémoir sur la Dispersion*, Paris, 1830; which is also repeated in the 1st livraison of the *Nouveaux Exercices*, Prague, 1835.

141. M. Cauchy commences by establishing the general equations as above (59) though in a different notation. In proceeding to integrate them he assumes ξ, η, ζ as each expressed in trigonometrical functions of x, y, z, while ρ is the resultant displacement perpendicular to a given plane, which determines the front of the wave. Then having taken the increments $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ as above, he introduces the hypothesis of symmetrical distribution by which those terms of $\Delta\xi$, &c. which involve $\sin 2\theta$ disappear; but not those involving $\sin^2\theta$. The equations (*Eq.* 25, *Memoir*) being thus reduced, a straight line is taken forming at the origin, with the axes, angles assumed in a certain relation to the coefficients of the equation, and involving a constant s^2, which has three values: shewing that there are three straight lines at right angles which fulfil the conditions, and these coincide with the semiaxes of a surface of the second degree to which as a subsidiary construction he refers in all his theory.

142. The application depends on a general property of such surfaces (established in the *Ex. Math.* Vol. III.

liv^{n}. 25), viz. that taking the general equation referred to the center the relation between the constants, expressed by s^2 is found by a cubic equation; it has therefore three values corresponding to the roots of that equation; if these are real, the surface is an ellipsoid, and the values of s^2 are the reciprocals of the semiaxes.

143. The angles formed by one such line with the axes being X, Y, Z, the displacement in that line being ȣ,

$$\text{ȣ} = \xi \cos X + \eta \cos Y + \zeta \cos Z,$$

and from the nature of the relation which gives s^2 (just mentioned) these results:

$$\frac{d^2\,\text{ȣ}}{dt^2} = -s^2\,\text{ȣ},$$

and the values of ξ, η, ζ being expressed each in terms of the three displacements ȣ$'$, ȣ$''$, ȣ$'''$, the solution is obtained for any one in terms of ρ in the form for two sets of waves corresponding to positive and negative values. Hence on recurring to the values of s^2 in terms of the original coefficients, we find it involve the relation which expresses the dispersion.

This process is manifestly equivalent to transforming the expressions for the displacements ξ, η, ζ (in the directions x, y, z) to other new components ȣ$'$, ȣ$''$, ȣ$'''$ in the directions of the new axes which are identical with those of the ellipsoid before mentioned.

SECTION V.

AXES OF ELASTICITY, WAVE SURFACE, &c.

AXES OF ELASTICITY.

144. It was before remarked that the conditions for the evanescence of certain terms (110) were connected with more general views. These refer to the remarkable property belonging to all elastic media (constituted as above supposed) *in which the molecules are symmetrically arranged*, that there are at every point three rectangular axes, such that if they be taken as the axes of co-ordinates, the terms in question will destroy each other, or the general equations will be reduced to the forms before stated (129).

These are termed "axes of elasticity". They were originally investigated by Fresnel, who defines them as "trois directions rectangulaires suivant lesquelles tout pétit déplacement de ce, point, en changeant un pen les forces auxquelles il est soumis, produit une resultante totile dirignée dum la ligne même de son déplacement." (*Mém, sur la double refraction, Mem. Instit.* Tom VII. p. 94.)

145. He also shews that if on each of these and all other radii, portions be taken from the origin as the square roots of the elastic forces in these directions respectively, the locus of the extremities, gives what he terms the "surface of elasticity." This determines the velocity of propagation of the wave when the direction of its vibrations is given. For the velocity of undula-

tory propagation, in an elastic medium being as the square root of the elastic force, must be represented by the radius vector of the surface of elasticity in the direction of the vibrations. (See Professor Lloyd's *Report on Optics.* B. Assoc^{n}. 1834, p. 383.)

146. From what was observed before (143) it thus appears that M. Cauchy's process is equivalent to transferring to the axes of elasticity, as axes of co-ordinates though not explicitly described as such, nor indeed is any mention of such axes made in the "*Mémoir sur la Dispersion*," though they are discussed in other parts of the *Exercices de Math.* Vol. IV. and V.

147. The nature of the proof of the general proposition (144) is sufficiently pointed out by Sir J. Lubbock in his paper already referred to. (*Journal of Science*, Vol. XV. Nov. 1839). It is in fact precisely similar in principle to that above given in a more restricted case (110): and consists essentially in a transformation of co-ordinates so as to cause the terms in question to disappear from the equation. It is precisely analogous to the process for causing terms involving odd products of the variables to disappear from the general equation to surfaces of the second degree: and is in fact identical with the dynamical investigation of the three principal axes of rotation, when the primary expressions have been laid down as referring to the case of elasticity, and are seen to have the same forms as those to which, in a different sense, we refer in the dynamical investigation.

148. To demonstrate the general proposition, that every system of molecules constituted as before supposed, and *arranged symmetrically*, has at every point three axes, such that, referred to these the differential equations

shall take the form above, is manifestly the same as to prove that if we take the resultant displacement Δr, we shall have directly from the symmetrical arrangement as before,

$$\Sigma\{\psi r \sin(k\Delta r)\,\Delta x\Delta y\} = 0,$$

and it only remains to shew that we likewise have,

$$\Sigma\left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right)\Delta x\Delta y\right\} = 0,$$

$$\Sigma\left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right)\Delta x\Delta z\right\} = 0,$$

$$\Sigma\left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right)\Delta y\Delta z\right\} = 0,$$

and this, the author points out is done simply in the following mannner.

Let

$$\begin{aligned}
x &= ax' + by' + cz', & \Delta x &= a\Delta x' + b\Delta y' + c\Delta z', \\
y &= a'x' + b'y'' + c'z', & \Delta y &= a'\Delta x' + b'\Delta y' + c'\Delta z' \ldots (\mathrm{B}), \\
z &= a'x' + b''y' + c''z', & \Delta z &= a''\Delta x' + b''\Delta y' + c''\Delta z'. \\
a^2 + b^2 + c^2 &= 1, & aa'' + bb'' + cc'' &= 0, \\
a''^2 + b''^2 + c''^2 &= 1, & a'a'' + b'b'' + c'c'' &= 0 \ldots (\mathrm{C}), \\
a'^2 + b'^2 + c'^2 &= 1, & aa' + bb' + cc' &= 0.
\end{aligned}$$

Let

$$\Sigma\left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right)\Delta x'^2\right\} = D,$$

$$\Sigma\left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right)\Delta y'^2\right\} = E,$$

$$\Sigma\left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right)\Delta z'^2\right\} = F,$$

$$\Sigma \left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right) \Delta y' \Delta z'\right\} = D',$$

$$\Sigma \left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right) \Delta x' \Delta z'\right\} = E',$$

$$\Sigma \left\{\psi r \sin^2\left(\frac{k\Delta r}{2}\right) \Delta x' \Delta y'\right\} = F'.$$

Then the problem is a purely mathematical one; and the solution consists simply in shewing that the equations B and C may be satisfied simultaneously by real values of the nine quantities a, b, c, &c., and to determine them.

In fact, as the author observes, in consequence of the signification here assigned to the quantities D, E, F, D', E', F', the process will be verbatim, the same as that of Poisson for the three principal axes of rotation. (*Mém. de l'Acad.* Vol. XIV. p. 320).

149. The author further observes particularly that the three axes of elasticity are not necessarily parallel in passing from one point to another,—except in perfectly homogeneous media. He remarks that M. M. Cauchy and Fresnel do not seem to have noticed the consequences of this circumstance: which he finds to coincide with the conditions the same as those which are supposed by M. Biot (*Mém. Acad.* Vol. XIII. p. 41) for his view of circular polarization.

WAVE-SURFACE.

150. These axes of elasticity are further to be considered in connexion with the *wave-surface*. It is to this subject that Sir J. Lubbock's remarks are mainly directed in the paper before quoted.

His chief object is generally to elucidate Fresnel's reasoning, which is involved in some obscurity; and to facilitate the comparison of Fresnel's views with those of other mathematicians, especially M. Cauchy: which not having been attended to by previous writers some confusion and controversy has arisen.

But this paper more particularly claims our attention as, supplying directly the connecting steps between Fresnel's fundamental expression, and the differential equations of motion as above deduced: or in a word, connecting the wave-surface with dynamical principles.

151. The investigation of the wave surface in all its generality is clearly a most important portion of the analytical theory of light, as involving the most comprehensive view of the entire system of double refraction and polarization, as well as the singular phenomenon of conical-refraction, so highly curious in itself, and so memorable from its having been predicted from theory by Sir W. R. Hamilton, though wholly unlike anything which the range of optical facts had previously exhibited.

To go into the details of these investigations is beyond the design of the present tract—but both on account of the intrinsic importance of the subject and its connexion with the general principles above developed, I shall here offer a very brief sketch, tending to place in a connected order, for the assistance of the student, the chief points involved.

152. Fresnel's original investigation of the general wave surface, in his celebrated memoir on double refraction, depends mainly upon the establishment of a certain equation, which for reference I will call (a), (p. 130). This he deduces from the consideration that v^2 will be a maximum when, if the force be resolved into two com-

ponents, (1) in the direction of the displacement, (2) in a direction perpendicular to it, the second component is also perpendicular to the plane of the wave; and the direction of the displacement remains unaltered.

Sir J. Lubbock shews that this property of the maximum is merely incidental, and that the equation (α) may be deduced directly from the first principles of the theory.

153. Adopting then the expressions above given, (138) for the cosines of the inclination of the resultant or line of force on the axes;

154. If this resultant be parallel to a plane

$$l'x + m'y + n'z = 0.$$

155. We thus have also,

$$a^2 l' \cos X + b^2 m' \cos Y + c^2 n' \cos Z = 0.$$

156. Again, the equation to the plane of the wave being

$$lx + my + nz = 0.$$

157. The direction of the displacement being parallel to it, we have also

$$l \cos X + m \cos Y + n \cos Z = 0.$$

158. If further the displacement be also parallel to the plane (154),

$$l' \cos X + m' \cos Y + n' \cos Z = 0.$$

159. And if the two planes (154) and (156) be perpendicular to each other, then also,

$$ll' + mm' + nn' = 0.$$

160. From these primary conditions the author easily obtains the equation

$$\left.\begin{aligned} &a^2 \cos X (m \cos Z - n \cos Y) \\ &+ b^2 \cos Y (n \cos X - l \cos Z) \\ &+ c^2 \cos Z (l \cos Y - m \cos X) \end{aligned}\right\} = 0,$$

which is the same as Fresnel's Eq. (A) *Mém.* p. 113), and Mr Sylvester's Eq. (6), (*Journal of Science*, 1837, Vol. XI. p. 463), (l, m, n, being the same as in Mr A. Smith's paper.)

Hence by eliminating the cosines the author arrives easily at the principal equation (a), *in the form into which Mr A. Smith puts it*, which will presently be given.

It will be observed that nothing here depends on the particular value assigned to a, b, c; but the whole deduction from the theory turns upon the expressions for the cosines of inclination of the resultant, (138): and we observed in the last section (139) that exactly similar expressions resulted from the complete theory as from Fresnel's. Thus the present investigation applies equally well to the more general theory: and the same remark may be extended to what follows.

161. Fresnel, though he gave the deduction of the fundamental equation (a) from the conditions of the vibrations, yet omitted the process by which he advanced from that equation to that of the wave surface, because, as he states, the calculations by which he assured himself of its truth were too tedious and embarrassing to insert. (*Mém.* p. 136).

These steps were supplied by M. Ampére (*Ann. de Chim.* Tom. XXXIX. p. 113), but by a very difficult and complex analysis.

Mr Archibald Smith, in a masterly paper in the Transactions of the Cambridge Phil. Society, (Vol. VI. p. 85, read May 18, 1835), has furnished the investigation in question in a most direct, brief, and elegant form.

162. Setting out from the equation (a) which in the form in which Fresnel gives it, is

$$\left.\begin{aligned} &(a^2 - v^2)(c^2 - v^2)\,n^2 \\ &+ (b^2 - v^2)(c^2 - v^2)\,m^2 \\ &+ (a^2 - v^2)(b^2 - v^2) \end{aligned}\right\} = 0.$$

To render it symmetrical, Mr A. Smith substitutes

$$m = -\frac{l}{n}, \qquad n = -\frac{m}{n}.$$

163. Whence it is easily brought into the form

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0.$$

164. Also if $l^2 + m^2 + n^2 = 1$.

165. Then $lx + my + nz = v$,

is the equation to a plane parallel to the plane of the wave at a distance v from the origin, whose value is found by Fresnel's equation.

166. Then differentiating these equations in respect to l, m, n, v, by eliminating the differentials and algebra, we at length obtain (writing $r^2 = x^2 + y^2 + z^2$),

$$\frac{a^2 x}{a^2 - r^2} = \frac{v^2}{r^2 - v^2}(lr^2 - xv),$$

and similar forms for b^2, y, m and c^2, z, n;

167. Whence

$$\frac{a^2 x}{a^2 - r^2} + \frac{b^2 y}{b^2 - r^2} + \frac{c^2 z}{c^2 - r^2} = 0.$$

168. And thence Fresnel's equation to the wave surface immediately follows; this is,

$$\left.\begin{aligned}&(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)\\&-a^2(b^2+c^2)x^2\\&-b^2(a^2+c^2)y^2\\&-c^2(a^2+b^2)z^2\\&+a^2b^2c^2\end{aligned}\right\}=0,$$

the axes of co-ordinates being throughout supposed coincident with those of elasticity.

169. By putting the equation (160) into a different form Sir J. Lubbock derives another equation employed in Mr Sylvester's investigation: whence again result two other forms, the same as Mr Sylvester's equations (c) and (d) (p. 464), whence again the same relations are deducible.

170. M. Cauchy's investigation of the wave surface is contained in his *Exercices de Math.* Vol. v. liv. 50, 51.) In previous Memoirs (Vols. III. and IV.) he had established general equations of motion for the molecules of elastic systems under different conditions of elasticity: and in the present Volume, in § 1, (p. 19) he applies these principles to light considered as homogeneous. He considers the several cases here explicitly referring to axes of elasticity (p. 25). Where the elasticity is the same in all directions, the general equations coincide with those originally deduced by M. Navier.

171. In § 2, he deduces the propagation of "plane waves" in three distinct sets, as above mentioned, and hence comes to the general conception of the wave surface. This he views as the "point de rencontre" of an assemblage of plane waves, having their fronts slightly inclined to each other and superposed at this point;—which "displaces itself" along a certain line with a velocity

different from that of the propagation of the waves. The co-ordinates of this point being x, y, z at the end of a time t the function

$$\Pi\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right) = 0,$$

represents the wave surface, which is touched by all the planes of the waves; there being three sheets corresponding to the three sets of plane waves, or values of s. From certain simple relations which arise he finds that as its radius r′ increases, the surface always continues similar to itself. (p. 35).

172. He also considers another curve surface closely related to the wave surface, represented by

$$F(x, y, z, t) = 0,$$

whose radius vector is r.

173. And from the relations of its co-ordinates to the plane of the waves he deduces its characteristic property to be, that if on its radii, distances are taken

$$\mathrm{r}'' = \frac{t^2}{\mathrm{r}},$$

the planes perpendicular to the extremities of these radii will be *tangent planes to the wave surface.*

174. We also have,

$$\mathrm{r} = \frac{t}{s},$$

other general relations are also established; by the aid of which he subsequently deduces the actual equation to the wave surface, involving as constants the same quantities which were concerned in the general equations given in § 1.

175. The equation is the following:

$$0 = \begin{cases} (x^2 + y^2 + z^2)(Px^2 + Qy^2 + Rz^2), \\ - t^2 P(Q + R)x^2, \\ - t^2 Q(R + P)y^2, \\ - t^2 R(P + Q)z^2, \\ + t^4 PQR. \end{cases}$$

176. He shews that the sections of this surface with the co-ordinate planes will be three circles and three ellipses represented by

$$\frac{y^2 + z^2}{P} = t^2, \qquad \frac{y^2}{R} + \frac{z^2}{Q} = t^2,$$

$$\frac{z^2 + x^2}{Q} = t^2, \qquad \frac{z^2}{P} + \frac{x^2}{R} = t^2,$$

$$\frac{x^2 + y^2}{R} = t^2, \qquad \frac{x^2}{Q} + \frac{y^2}{P} = t^2.$$

177. This equation is of the same form with that of Fresnel, differing only in the circumstance that all the terms except the first are multiplied by some power of t; it can therefore only be identical with Fresnel's when $t = 1$, or embraces it as a particular case under that condition.

In other words, it may be considered as identical with Fresnel's surface, if the latter be regarded as only expressing the form of the wave propagated during an unit of time.

178. In this case however we must observe that other consequences are involved: for if we make $t = 1$ we have in the equation (174)

$$s = \frac{1}{\mathrm{r}},$$

that is, in this case for each of the three values of s the values of r must be constant.

179. The constants P, Q, R, which enter this equation to the wave surface are quantities which result from the original investigation of the equations of motion; in which M. Cauchy introduces the values of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ in forms corresponding to (21), viz.,

$$\Delta\xi = -2\sin^2\theta\,\xi - \sin^2\theta\,\frac{d\xi}{dx}, \text{ \&c.}$$

and on the principle of the symmetrical arrangement, omitting the quantities multiplied by the second term of this increment, the equations of motion appear in the form

$$\frac{d^2\xi}{dt^2} = \mathrm{L}\xi + \mathrm{R}\eta = \mathrm{Q}\zeta,$$

$$\text{\&c.} = \text{\&c.};$$

where in my notation,

$$\mathrm{L} = \Sigma 2\left[m\left\{\phi(r) + \psi(r)\,\Delta x^2\right\}\sin^2\theta\right],$$

$$\mathrm{R} = \Sigma 2\left[m\left\{\psi(r)\,\Delta x\,\Delta y\right\}\sin^2\theta\right],$$

$$\mathrm{Q} = \Sigma 2\left[m\left\{\psi(r)\,\Delta x\,\Delta z\right\}\sin^2\theta\right], \text{ \&c.}$$

Also M. Cauchy uses the letters a, b, c for the constants in the equation to the plane of the wave, or corresponding to l, m, n above.

180. In his investigation of the wave surface (*Exerc.* liv. 50, 51) he traces the modifications which belong to the several cases of elasticity: and it appears that in the general case of three axes of elasticity (p. 26) the above coefficients are expressed in terms of certain other quantities together with a, b, c, thus,

$$\mathrm{L} = La^2 + Rb^2 + Qc^2,$$
$$\mathrm{M} = Ra^2 + Mb^2 + Pc^2,$$
$$\mathrm{N} = Qa^2 + Pb^2 + Nc^2,$$
$$\mathrm{P} = 2Pbc, \quad \mathrm{Q} = 2Qca, \quad \mathrm{R} = 2Rab.$$

And further, in the immediate deduction of the wave surface (p. 61, &c.) by a refined analysis, the author shews that on neglecting infinitesimals we have certain approximate relations between these constants, L, M, N, and P, Q, R.

It is thus that the quantities L, M, N disappear from the expressions, and the resulting equation involves only P, Q, R. And from the relations thus subsisting with the original quantities L, M, N, &c. the coefficients P, Q, R each involve as factors both Δx^2 and Δx, Δy, &c. They therefore differ from the coefficients a, b, c, before employed, which involve only Δx^2 as above shewn. In fact, the circumstance of the terms involving $q \sin^2 \theta$ being retained in the differential equations influences materially the whole form of the investigation.

CUSPS AND CONICAL REFRACTION.

181. One of the most remarkable properties of the wave surface is that connected with the existence of *cusps*, which may be thus deduced:

Calling the equation to the wave surface V, by the principles of analytical geometry, the cosines of the inclinations of the tangent plane with the co-ordinate planes will be

$$\frac{\dfrac{dV}{dx}}{\sqrt{\left\{\left(\dfrac{dV}{dx}\right)^2 + \left(\dfrac{dV}{dy}\right)^2 + \left(\dfrac{dV}{dz}\right)^2\right\}}},$$

and the like for the other co-ordinates.

182. Now we have on differentiating the equation,

$$\frac{dV}{dx} = \begin{cases} 2x\,(a^2x^2 + b^2y^2 + c^2z^2), \\ + 2\,(x^2 + y^2 + z^2)\,a^2x, \\ - 2\,a^2\,(b^2 + c^2)\,x. \end{cases}$$

183. If in this expression we suppose $y = 0$, it becomes

$$= \begin{cases} 2x\,(a^2x^2 + c^2z^2), \\ + 2\,(x^2 + z^2)\,a^2x, \\ - 2a^2\,(b^2 + c^2)\,x. \end{cases}$$

184. Again, if $y = 0$, we have also manifestly,

$$\frac{dV}{dy} = 0.$$

185. And again, with the same condition, we shall similarly have

$$\frac{dV}{dz} = \begin{cases} 2z\,(a^2x^2 + c^2z^2), \\ + 2\,(x^2 + z^2)\,c^2z, \\ - 2c^2\,(a^2 + b^2)\,z. \end{cases}$$

186. Also when $y = 0$ it is easily seen that the equation may be put in the form

$$(x^2 + z^2 - b^2)\,(a^2x^2 + c^2z^2 - a^2c^2) = 0\,;$$

or it is resolved into two factors which give

$$x^2 + z^2 = b^2,$$

$$a^2x^2 + c^2z^2 = a^2c^2.$$

187. Whence we obtain

$$z = \pm a \sqrt{\frac{b^2 - c^2}{a^2 - c^2}},$$

$$x = \pm c \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}.$$

But in substituting the values (186) in the equations (183, 185) it thus appears that for these values of x and z, and $y = 0$, we have

$$\frac{dV}{dx} = 0, \quad \frac{dV}{dy} = 0, \quad \frac{dV}{dz} = 0,$$

and consequently each cosine of inclination $= \frac{0}{0}$; as we suppose $a > b > c$, these values of x and z are rational: and, as we have two values of each, it follows that there are four points in the plane of xz determined by these values, at which the cosines of inclination of the tangent plane become each indeterminate, or to which an infinity of tangents belong, which is manifestly the characteristic of a *conoidal cusp*, having a *tangent cone.*

188. The two equations (186) are manifestly those of an ellipse and a circle, which form the section of the wave surface in the plane xz; (b) being the radius of the circle, and (a) and (c) the semi-axes of the ellipse. They intersect in four points in this plane, where the sections of the cusps appear by the intersections of the curves.

The angle at which they intersect, or the vertical angle of the cone may be easily found.

189. It further appears from the section in the plane of xz that the same line is a tangent both to the ellipse, and to the circle, at a point in each near their point of

intersection. These points lie in what constitutes the margin of the conoidal cusp, when we refer to the general surface; and it can be shewn that the same thing holds good all round this margin: or that the plane of which the tangent in question is a section, is a tangent plane to the wave surface in an infinity of points, lying in all directions round the margin of the cusp; or giving a circle of contact.

190. To follow out these deductions, and to pursue the subject of the cusps in all its generality, the student must have recourse to the investigations of Sir W. R. Hamilton, who, in the third supplement to his profound memoir, on Systems of Rays (*Mem. R. I. Acad.* Vol. XVII. p. 128), has deduced the wave surface from his own principles and thence originally pointed out the deduction of the cusps, and predicted the optical phenomenon which ought to result.

191. Having the wave surface equation (which he expressly calls the "curved unit-wave") in the same form as Fresnel's; he changes it to polar co-ordinates; and thus obtains a polar equation to the wave surface in the same form as Fresnel: and thence by a transformation to other co-ordinates (making in fact the radius (ρ') through the cusp, the axis of z) he gets the equation into such a form that it readily shews the nature of the surface in the neighbourhood of the cusp: in other words, for the point coinciding with the cusp, the equation becomes that of a conical surface, which is therefore the tangent to the surface at this point.

Fresnel had deduced the polar equation in a similar form, from which he observed that it gives in general

two unequal values of the radius, or velocity, except for two particular directions determined as above, which he terms optic axes: but Sir W. R. Hamilton assigns them the more expressive name of "lines of single ray velocity." Their inclinations to the axes are assigned by expressions deduced from those above given, viz.:

$$\cos \rho' x = \frac{c}{b} \sqrt{\frac{a^2 - b^2}{a^2 - c^2}},$$

$$\cos \rho' z = \frac{a}{b} \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

On somewhat similar considerations Fresnel had also noticed, that any given normal direction corresponds in general to two unequal *normal* velocities, except in the direction of two rays, which Sir W. R. Hamilton therefore designates as "lines of single normal velocity."

The inclinations to the axes of these rays, which are those perpendicular to the common tangent of the ellipse and circle in the plane of $x z$, are analogous to the former, and are assigned (from the geometry of the ellipse and circle) by the expressions,

$$\cos \omega' x = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}},$$

$$\cos \omega' z = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

Here again transferring to new axes, and taking the line ω' as the axis of z, Sir W. R. Hamilton brings the wave surface equation into another form, which enables him to shew, that for the value $z_{,,} = b$ it becomes the equation of a circle, which is the *circle of contact* already spoken of, and whose magnitude is thus given, and consequently the direction of the rays undergoing this peculiar refraction, determined.

192. Some valuable illustrations of the nature of the wave surface, its cusps, &c. will be found in the constructions given by Prof. Maccullagh, entitled "Geometrical Propositions applied to the Wave Theory," &c. (*Mem. R. I. Acad.* Vol. XVII.); more especially a very concise deduction of the equation to the wave surface from the generating ellipsoid in a note at the end of the paper. Some other important elucidations are given by Prof. Sylvester: (*Journal of Science*, Vol. XII. p. 78) and in Prof. Young's *Theory of Curved Surfaces, &c.* p. 165.

193. The application of this part of the theory has been given by Prof. Lloyd (*Mem. R. I. Acad.* Vol. XVII.) who subjected it to experimental examination at the request of Sir W. R. Hamilton: a copious abstract of the investigation also appeared in two papers in the *Journal of Science*, 1833, Vol. II. p. 112 and 207.

194. The physical phenomenon corresponding to the geometrical fact of the conoidal cusps is termed conical refraction.

In a biaxial crystal in which the double refractive force is of sufficient energy to render the effect sensible, a ray passing through the crystal in the direction of its optic axis, will emerge at a cusp: that is, will have no longer one direction, but an infinite number, arranged in a conical surface round the axis, or will have a luminous ring for its section if received on a screen.

195. Again, we have seen that the portion of the surface round the margin of the cusp will have a common tangent plane touching it in a circle. Hence parallel rays incident on this circle will all converge to a point within the crystal, forming another conical surface,

or conversely, a ray so refracted will at emergence form a *cylindrical* luminous surface.

The former kind is called external and the latter internal conical refraction.

196. Prof. Lloyd describes in detail the means he took for verifying both cases by actual observation. He chose a crystal of Aragonite as in most respects best suited to give conspicuous results, and having ascertained the *general* facts, he proceeded to examine the magnitude and position of the cones in the respective cases. These are determined from theory by the formulas given above, and were found to agree as closely as could be expected with the measures of the observed phenomena.

OTHER RESEARCHES CONNECTED WITH THE FOREGOING.

197. I have before alluded to the material question, whether in plane polarized light the vibrations take place *in* the plane of polarization or perpendicular to it; which has been so differently viewed by several eminent mathematicians.

This question is closely connected with the investigation of the curve surface. The ordinary and extraordinary rays are, respectively, those produced by vibrations belonging to the several sheets of the wave surface: and the expressions deduced for them indicate in which direction the ordinary and extraordinary vibrations are performed relatively to the plane of the refraction. In this respect then, the results of the principles adopted by the mathematicians on either side are brought into comparison and their truth put to the test.

Mr Tovey agrees in the view of Prof. Maccullagh, and has deduced expressions of this kind (See *Journal of Science*, Vol. IX. p. 428, and Vol. XI. p. 422, &c.) from

whence he finds that, in the case of an uniaxial crystal, when the vibrations are perpendicular to the axis of the crystal they belong to the extraordinary ray, or to the spheroidal wave: whereas in the corresponding case, according to Fresnel's expression, they belong to the ordinary ray, or to the spherical wave.

To discuss this subject more fully would be altogether beyond my present design and limits. But to those who are following up the analysis of the theory it must, I think, appear peculiarly important to examine into the nature of this apparent contradiction, and to ascertain its origin, and to what extent it involves the principles previously adopted.

198. Related also to the main subject, but which must, for similar reasons, be dismissed with merely a brief allusion, is the discussion of the nature of the rays in quartz. On this subject, Mr Tovey's paper (*Journal of Science, May*, 1839, Vol. XIV. p. 323.) is eminently deserving of examination, in connexion, especially with the general view of elliptic polarization considered above, as also with Professor Maccullagh's valuable paper in the Memoirs of the Royal Irish Academy, 1836, in which he connects the laws of M. Biot, and the theory of Mr Airy, with certain differential equations of vibratory motion, provided the quantities α, β, n, and k, are so assumed as to fulfil these conditions, viz. that if A and B be the squares of the velocities of the ordinary and extraordinary rays in common double refraction, in quartz, we replace them respectively by

$$A - k\frac{\beta}{\alpha}C, \text{ and } B - k\frac{\alpha}{\beta}C,$$

where C is a new constant determined from Biot's observations.

199. The equations referred to are these:

$$\left.\begin{aligned}\frac{d^2\eta}{dt^2} &= A\frac{d^2\eta}{dx^2} + C\frac{d^3\zeta}{dx^3}\\ \frac{d^2\zeta}{dt^2} &= B\frac{d^2\zeta}{dx^2} - C\frac{d^3\eta}{dx^3}\end{aligned}\right\}.$$

But on taking the partial differential coefficients of the expressions for elliptical vibrations (40) it is easily seen (from the particular forms of those functions) that the above equations are reducible to,

$$\left.\begin{aligned}\frac{d^2\eta}{dt^2} &= (A - k\frac{\beta}{\alpha}C)\frac{d^2\eta}{dx^2}\\ \frac{d^2\zeta}{dt^2} &= (B + k\frac{\alpha}{\beta}C)\frac{d^2\zeta}{dx^2}\end{aligned}\right\}$$

which are of the form for vibratory motion.

200. The addition of these new terms in the differential equations was however in the first instance hypothetical. The author has since pursued an additional investigation, in which he assigns at least a very probable dynamical explanation of the origin of these additional terms. (See proceedings of the *Royal Irish Academy*, No. 20, p. 385.)

I can here, however, do no more than recommend these profound investigations to the student's attention.

201. This may be also the place to allude (though it must be only in the most cursory manner) to some other important investigations connected with the proceeding, which must especially claim the attention of the student.

202. Mr Tovey, in following out some ideas arising from a former investigation, has developed (*Journal of*

Science, Vol. xv. p. 450, 1839) a more general mode of deducing the integral of the differential equations (64) by supposing the functions of x and t which constitute that solution, expressed in logarithmic forms involving certain imaginary terms; whence, by several transformations, he obtains the solution in the form,

$$\eta = \Sigma \left\{ a e^{\epsilon x} \sin (nt + kx + b) \right\};$$

whence from the value assignable to ϵ as dependent upon the imaginary terms, the author shews that a *general* theory at least is established, explaining how absorption depends on the thickness of the medium traversed, and on the wave-length jointly.

203. Some highly valuable researches bearing upon the entire theory of the propagation of light, have been communicated to the Royal Irish Academy by Professor Lloyd, of which abstracts appear in the proceedings of the Academy (No. i. p. 10, and No. ii. p. 25. Nov. 1836.) But I believe the paper has not yet been published in detail.

204. In the first instance the author introduces certain improvements upon the method of M. Cauchy, by which he obtains not only the formulas already referred to, but also in a more general case an expression for the displacement exactly similar to Mr Tovey's, though deduced on different principles; that is, involving a similar variable exponent: from which the author observes, it follows that the amplitude of the displacement, and therefore the intensity of the light decreases in geometrical progression, as the distance increases in arithmetical: and as the term corresponding to ϵ in Mr Tovey's formula is a general function of n or of the

colour, the differently-coloured rays will be *differently absorbed*; the complete value of the displacement being the sum of a series of like terms, this gives a general explanation of the absorption; since it is only necessary to admit that the function above mentioned varies in certain cases *rapidly* with moderate changes in n, and becomes *very great* for certain definite values of that quantity.

205. Besides this the author has entered into elaborate investigations connected with the law of dispersion: more particularly in reference to that part of the investigation in which, following the method of M. Cauchy, the *sums* are converted into *definite integrals.* In this point of view the coefficients of the dispersion series occupy his attention, and, *as thus deduced*, he finds them result with values inconsistent with each other. He follows up the subject by attempting to remove the difficulty by taking into account the action of the material molecules of the medium.

206. It does not appear, however, that this objection will affect the dispersion formula as above considered; it being deduced independently of this latter process: I referred in one of my former papers to Professor Lloyd's objection under a mistaken impression that it did apply to the ordinary formula.

207. But the whole subject seems likely to undergo an almost entire revolution in its first principles, when the proposed researches of Sir W. R. Hamilton shall be completed. These investigations are conducted by an analysis of the highest order, of which a short account is given in the proceedings of the *Royal Irish Academy*, No. xviii. p. 341.

As it would be impossible here to enter upon such a subject, it must suffice to mention that in its ulterior results, this theory agrees with the common views in leading to the *same expression for the dispersion:* while this is intimately connected with another formula, indicating a new and highly curious result relative to the velocity with which the vibrations are communicated to the contiguous parts of the ætherial medium. (*Ib.* p. 349).

SECTION VI.

APPLICATION OF THEORY TO THE PHENOMENA OF DISPERSION.

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208. Having before deduced the theoretical expression for the general relation between the index and the wave-length for a given ray in a given medium, I proceed in the present section to consider more in detail the reduction of this expression into forms adapted to calculation, and to examine more particularly the degree of accordance thus to be discovered between the theory and the numerical data furnished by prismatic observations, as well as the general sufficiency of the principle on which the methods of calculation are framed.

DEDUCTION OF THE FORMULA.

209. The formula for the dispersion at first obtained, viz.

$$\frac{1}{\mu^2} = \Sigma \left\{ H^2 \frac{\sin^2\left(\frac{\pi\Delta x}{\lambda}\right)}{\left(\frac{\pi\Delta x}{\lambda}\right)^2} \right\},$$

on developing the sine and dividing by the arc will give

$$\frac{1}{\mu^2} = \Sigma \left[H^2 \left\{ 1 - \frac{(\pi\Delta x)^2}{6\lambda^2} + \frac{(\pi\Delta x)^4}{120\lambda^4} - \&c. \right\}^2 \right]$$

$$= \Sigma \left[H^2 \left\{ 1 - \frac{(\pi\Delta x)^2}{3\lambda^2} + \frac{2(\pi\Delta x)^4}{45\lambda^4} - \&c. \right\} \right].$$

210. This is easily seen to be identical with the formula of Mr Kelland, viz.

$$\frac{1}{\mu^2} = p - \frac{q}{\lambda^2} + \frac{l}{\lambda^4}.$$

211. And with that which arises directly in the integration according to the mode pursued by Mr Tovey and Sir J. Lubbock: viz.

$$n^2 = h_1 k^2 - h_2 k^4 + \&c.$$

which gives directly on dividing by k^2,

$$\frac{1}{\mu^2} = h_1 - h_2 \frac{(2\pi)^2}{\lambda^2} + \&c.$$

212. These again are identical with the formula (more generally expressed) of M. Cauchy, deduced in the (*Nouveaux Exercices*, liv^n. 2.) first in the equation (80), (p. 40), and thence in equation (36), (p. 56), which last formula is,

$$\Omega^2 = a_1 + a_2 k^2 + a_3 k^4 + \&c.$$

where $\Omega = \frac{1}{\mu}$ and k has the same value as above.

INFERENCES FROM THE FORMULA.

213. We will first observe that if Δx be very small compared with λ we shall have for the term on which the dispersion depends, very nearly,

$$\frac{\sin^2\left(\frac{\pi \Delta x}{\lambda}\right)}{\left(\frac{\pi \Delta x}{\lambda}\right)^2} = 1.$$

It follows, then, with regard to the hypothetical nature of the æthereal medium, that if the interval between two molecules be *very* much less than the length of a

wave, then the velocity will not sensibly vary with the length of a wave, or no dispersion will take place.

214. In adopting the theory, then, with a view to its application to the facts, we must carefully observe the limitation thus imposed upon the primary nature of our hypothesis. It is a limitation which is perfectly admissible as regards any of the preceding deductions; and we must introduce it as an express condition, *that a relation between the velocity and the length of a wave is established on M. Cauchy's principles*, provided *the molecules are so disposed that the intervals between them always bear a sensible ratio to the length of an undulation.*

This condition was suggested by Mr Airy: it is not explicitly stated by M. Cauchy, (in his *Mem. on Dispersion*) though it is implied in his subsequent applications of the formula in the *Nouv. Ex.* liv^{n}. 2.

215. The dispersion is insensible in free space, and large in dense media. Hence Mr Kelland has inferred that the æther is in a state of *greater* density in free space, and of *less* density in dense media.

216. Here another inference of considerable interest suggests itself: in the case supposed above (213) the formula (209) is reduced to

$$\frac{1}{\mu^2} = \Sigma \,.\, H^2,$$

or the several series are reduced to their first term.

217. Now, in any medium, if Δx be *not* very small compared with λ, we shall have different refractive indices for each ray, which will differ less as λ becomes greater: and if we suppose λ to increase indefinitely, (Δx remain-

ing finite) the above expression (216) will be the *limit* to which the refractive index constantly tends, and from which it does not sensibly differ when λ is supposed large; that is, for some ray beyond the red end of the spectrum (and such may exist though not sensible to the eye) there is *an absolute limit to all refraction*, different in different media; if we had any means of determining it by experiment, it would be a most important step for the theory. This quantity corresponds to (h) in Mr Tovey's notation, and to (p) in Mr Kelland's who has determined it for Fraunhofer's media; as I have done in a different way, both for these media and those examined by M. Rudberg; where this quantity appears as the empirical constant: *Phil. Trans.* 1835, Pt. I. and 1836, Pt. I. It is an index not greatly below that for the red extremity of the visible spectrum. Whether it may be connected with the refraction of *heat* becomes an interesting question, to which we shall recur.

METHODS OF CALCULATION.

APPROXIMATE FORMULA.

218. The formula above given, it may be well to bear in mind, is equivalent to the sum of a number of like terms, for the same value of μ and of λ, involving different values of H and Δx, and the summation extending to all the values of Δx which it may be necessary to include.

219. The formula for calculation which I at first adopted, was deduced by taking *a single term* of the above form with some constant coefficient which might be a sort of mean among all the similar terms and which on extracting the root would give,

$$\frac{1}{\mu} = H\left\{\frac{\sin\left(\frac{\pi\Delta x}{\lambda}\right)}{\left(\frac{\pi\Delta x}{\lambda}\right)}\right\}.$$

The practical method of applying it consisted in finding in the first instance, by a tentative process, (equivalent to assuming the two extreme indices from observation) a fundamental arc θ, such as should accord with the two extreme indices as closely as possible, when reduced in the ratio of the wave-lengths, and the ratio of the arc to the sine multiplied by a constant: from such a value the others were found by a like reduction in the ratio of the wave-lengths.

220. By this method I made all the calculations in my first two papers in the *Philosophical Transactions* (1835, Pt. I. and 1836, Pt. I.), in which I compared with theory all the indices furnished by the observations of Fraunhofer and Rudberg.

But this method was indirect, and several better were soon suggested, one such was stated in my paper in the *Journal of Science*, Vol. viii. p. 309, which applies to all cases in which it may be deemed sufficient to take into account only two terms of the series.

221. It is easy to shew the degree of approximation here attained by comparison with the *exact* formula, as follows:

Taking the series as before, (209)

$$\frac{\sin^2\theta}{\theta^2} = 1 - \frac{\theta^2}{3} + \frac{2\theta^4}{45} - \&c.$$

The formula on the hypothesis of one constant coefficient will thus be,

$$\frac{1}{\mu^2} = H_1^2 - \frac{1}{3} H_1^2 \left(\frac{\pi}{\lambda}\right)^2 \Delta x^2 + \frac{2}{45} H_1^2 \left(\frac{\pi}{\lambda}\right)^4 \Delta x^4 - \&c.\,;$$

while the exact formula will give,

$$\frac{1}{\mu^2} = \Sigma H^2 - \frac{1}{3} \left(\frac{\pi}{\lambda}\right)^2 \Sigma (H^2 \Delta x^2) + \frac{2}{45} \left(\frac{\pi}{\lambda}\right)^4 \Sigma (H^2 \Delta x^4) - \&c.$$

To compare these, let

$$H_1^2 = \Sigma H^2 \text{ and } \theta = \frac{\pi}{\lambda} \sqrt{\frac{\Sigma (H^2 \Delta x^2)}{\Sigma H^2}};$$

Then on substituting,

$$\frac{1}{\mu^2} = \Sigma H^2 - \frac{1}{3}\left(\frac{\pi}{\lambda}\right)^2 \Sigma (H^2 \Delta x^2) + \frac{2}{45} \Sigma H^2 \left(\frac{\pi}{\lambda}\right)^4 \frac{\{\Sigma (H^2 \Delta x^2)\}^2}{(\Sigma H)^2} - \&c.$$

These are identical in their two first terms: and may possibly differ but little in the third.

222. The difference between these formulas is manifestly that which is involved in the summation of the variable terms, and would disappear if we could assume

$$\Sigma (H^2 \Delta x^2) = (\Sigma H^2)(\Delta x^2),$$

$$\text{or} \qquad = H_1^2 (\Delta x^2),$$

or since the values of Δx are the same for all the molecules situated in a plane parallel to yz at the same distance in x, this would be equivalent to taking into account only two adjacent parallel strata of molecules. As an approximate supposition this may be conceived justifiable; at all events it is certain, that such a formula affords the closest accordance with the truth throughout the whole range of low-dispersive substances: and even among many of the higher it gives a very near approach to such agreement.

223. But a more exact method of calculation by the same formula, that is, taking three terms into account, I afterwards gave in the same *Journal* (Vol. XIV. April, 1839).

This method, with a slight improvement in its form, which renders its application easier in actual computation, is as follows:

For any one ray

$$\frac{1}{\mu} = H\left\{1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right\}.$$

For any other ray, reducing in the ratio of the wave-lengths,

$$\frac{1}{\mu_{,}} = H\left\{1 - \frac{1}{6}\theta^2\left(\frac{\lambda\mu_{,}}{\lambda_{,}\mu}\right)^2 + \frac{1}{120}\theta^4\left(\frac{\lambda\mu_{,}}{\lambda_{,}\mu}\right)^4\right\},$$

whence we have

$$\frac{1}{H} = \mu - \frac{\mu\theta^2}{6} + \frac{\mu\theta^4}{120},$$

$$\frac{1}{H} = \mu_{,} - \frac{\mu_{,}}{6}\theta^2\left(\frac{\lambda\mu_{,}}{\lambda_{,}\mu}\right)^2 + \frac{\mu_{,}}{120}\theta^4\left(\frac{\lambda\mu_{,}}{\lambda_{,}\mu}\right)^4\Bigg\},$$

and eliminating H,

$$0 = \begin{cases} (\mu_{,} - \mu) - \dfrac{1}{6}\left\{\mu_{,}\left(\dfrac{\lambda\mu_{,}}{\lambda_{,}\mu}\right)^2 - \mu\right\}\theta^2 \\ \qquad + \dfrac{1}{120}\left\{\mu_{,}\left(\dfrac{\lambda\mu_{,}}{\lambda_{,}\mu}\right)^4 - \mu\right\}\theta^4; \end{cases}$$

or for brevity,

$$0 = m - h\theta^2 + k\theta^4;$$

whence by solving the quadratic,

$$\theta^2 = \frac{h}{2k} \pm \sqrt{-\frac{m}{k} + \frac{h^2}{4k^2}}.$$

224. By this method if we take the *two* values μ and $\mu_{,}$ from observation, we can find θ; or thence again

obtain the value of H for every other ray of the spectrum, which, if the formula be correct, ought to result the *same* for every ray.

This process then, (supposing we can consider it sufficiently sanctioned by theory), it should be particularly remarked, would require only the assumption of *two* values, as given by observation for each medium. Now though I have not as yet tried the result of calculation *precisely in the way* above stated, yet it is evident that all the calculations I made in my two first papers in the *Phil. Trans.*, including all the results of Fraunhofer and Rudberg, were conducted on an hypothesis which is *in principle identically the same:* and in all those it is universally allowed the coincidences are as close as could be desired: so that it is evident that this supposition cannot be very far from the truth; calculation grounded upon it has not yet been applied to more highly dispersive media. But it may be in several respects important to see whether it may be so applicable; whether it may apply even *as well* as the methods proposed on the less restricted hypotheses, such as I have employed in my later calculations.

EXACT FORMULA.

225. If we take the *exact* formula, for its practical application, we must limit the number of terms taken: and it may be readily allowed that the series converges fast enough to admit of the three first being considered sufficient.

Thus there are of necessity involved three coefficients neither the *values* of which, nor any *relation* between them, are assigned by theory. *Directly or indirectly*, then, they must be assumed from observation; that is, the pro-

blem will consist in finding *the value of any required index as interpolated between three others assumed.* For this purpose several methods have been proposed, and though in some measure different in their form, yet they are essentially equivalent in principle to such an interpolation.

M. CAUCHY'S METHOD.

226. M. Cauchy's process of calculation is given in his *Nouveaux Exercices*, liv^{n}. 2. Sec. 3—5. It is of great length and complexity, but the general principle may be very briefly stated as follows:

Having deduced the series for the index in even powers of the inverse wave-length, by reverting the series, he obtains a similar form for k^2 in even powers of n; then taking the successive powers of n from the former series, and substituting in the last, an identical equation results, from which the values of the coefficients are expressed in terms of those of the first series; they are seen to decrease rapidly in successive terms, by reference to the original form in which they appear as functions of r, the distance of the molecules, which is supposed very small. Thus if we retain successively 2, 3, &c. terms of the series we can eliminate as many of the coefficients. In this way successive approximations are made by an elaborate analysis. This is applied numerically in a series of tabular results of great length; the data assumed from observation are the sums of the squares of the indices for all the rays in the same medium, and for the same ray in all the media considered, and the mean indices respectively deduced. By applying to these data the formulas of theory as derived upon the supposition of taking four terms of the original series, there are at length obtained the theoretical squares of the indices; and on

comparing these with those of observation the differences are found insensibly small. The cases considered are those, *only, of the media observed by Fraunhofer.* The numerical computations extend to 118 pages 4to. occupying the whole of livraisons 3 to 6.

A supplementary calculation in liv^n. viii. sec. 11, is devoted to examining to what extent errors in the value of λ may affect the results, which is found to be quite within the limits of probable errors of observation.

Lastly, in sec. 12, M. Cauchy gives a shorter method of deducing the formula for a first approximation, which is nearly the same as that in his memoir presented to the Academy of Science in 1830, and published in Ferussac's Bulletin (tom. XIV. p. 9). He refers also to my deduction of a similar formula independently.

PROFESSOR KELLAND'S METHOD.

227. Professor Kelland's formula, as before stated, (expressed for the wave-length in the medium) is,

$$\frac{1}{\mu^2} = p - \frac{1}{\lambda_1^{\,2}} q + \frac{1}{\lambda_1^{\,4}} l,$$

or reducing to the wave-length in air, it becomes

$$\frac{1}{\mu^2} = p - \left(\frac{\mu}{\lambda}\right)^2 q + \left(\frac{\mu}{\lambda}\right)^4 l,$$

taking such formulas successively for the different standard rays, between any two, as those for B and E, the constant p is eliminated: and combining these with a third, as that for H, the coefficients q and l are determined. For brevity writing

$$\frac{1}{\mu^2_{\,B}} = b, \quad \left(\frac{\mu}{\lambda}\right)^2_B = \beta, \quad \left(\frac{\mu}{\lambda}\right)^4_B = \beta^2;$$

and similarly expressing by e, ϵ, ϵ^2; h, η, η^2; the corresponding quantities for the rays E and H, we shall have

$$(b - e) = (\epsilon - \beta)\, q - (\epsilon^2 - \beta^2)\, l,$$

$$(e - h) = (\eta - \epsilon)\, q - (\eta^2 - \epsilon^2)\, l;$$

whence we obtain,

$$l = \frac{(\epsilon - \beta)(e - h) - (\eta - \epsilon)(b - e)}{(\epsilon - \beta)(\eta^2 - \epsilon^2) - (\eta - \epsilon)(\epsilon^2 - \beta^2)},$$

$$q = \frac{(b - e) + (\epsilon^2 - \beta^2)\, l}{(\epsilon - \beta)}.$$

Having thus by the assumption of three indices determined the constants q and l for the medium, by the aid of these, combined again with the indices, a value of p is deduced for each ray, viz.

$$p = \frac{1}{\mu^2} + \left(\frac{\mu}{\lambda}\right)^2 q - \left(\frac{\mu}{\lambda}\right)^4 l,$$

and if these values derived from the different rays result *equal* the theory is verified.

228. The author has thus verified it to a degree which will probably be deemed amply sufficient for all the media examined by Fraunhofer; notwithstanding a small discrepancy which seems always to attach to the ray G.

229. In thus determining the value of p for each medium, it should be observed, Mr Kelland has furnished a datum which (as already observed) under another point of view is of great interest, viz. the *limit* of least refrangibility for these substances.

SIR W. R. HAMILTON'S METHOD.

230. The method proposed by Sir W. R. Hamilton is given at length in my paper in the *London and Edinburgh Journal of Science*, Vol. viii. March, 1836. It is deduced by taking the series for $\frac{1}{\mu^2}$ to three terms, with general coefficients; when reducing the values of λ in the medium to those in vacuo, each term of course involves μ. Substituting again for μ in each term its value derived from the series, expressions are obtained by which the original coefficients are eliminated, and certain others found which depend only on the wave-lengths in vacuo, and are thus independent of the medium.

These constants can be readily calculated for each ray, from the known values of λ as determined by Fraunhofer. And the process at length deduced, consists in assuming any three indices from observation for the particular medium, from which any one of the four remaining is found by introducing the constants just spoken of for those rays respectively.

231. More precisely;—in the series

$$\frac{1}{\mu^2} = A_0 - \frac{A_1\mu^2}{\lambda^2} + \frac{A_2\mu^4}{\lambda^4} - \&c.$$

The coefficients A_0, A_1, A_2, &c., are independent of the medium.

232. If we take a similar series, extract the root, and develope the reciprocal, we find,

$$\mu = a_0 + \frac{a_1}{\lambda^2} + \frac{a_2}{\lambda^4} + \&c.$$

Substituting this value for μ in the first series, and equating to zero the coefficients of like powers of λ, he finds the

values of a_0, a_1, a_2, in terms of A_0, A_1, A_2, or those coefficients are independent of μ or of the medium.

233. Hence if we take four such series for any four rays, as for μ_B, μ_D, μ_F, μ_H, we can eliminate between them all the three constants.

234. If we write $\frac{1}{\lambda} = l$, and designate the indices and wave-lengths for particular rays by subscribing the letters making those rays, the result of the elimination gives

$$0 = \begin{cases} \quad (\mu_D - \mu_B)\,(l^2_H - l^2_B)\,(l^2_F - l^2_B)\,(l^2_H - l^2_F), \\ -\,(\mu_F - \mu_B)\,(l^2_H - l^2_B)\,(l^2_D - l^2_B)\,(l^2_H - l^2_D), \\ +\,(\mu_H - \mu_B)\,(l^2_F - l^2_B)\,(l^2_D - l^2_B)\,(l^2_F - l^2_D). \end{cases}$$

Here the coefficients, involving only wave-lengths, might be calculated once for all, as applicable to all media: and thus having three indices assumed from observation any fourth could be found.

235. Practically, however, this method (though simpler in theory) is not found so convenient as a modification of it, which Sir W. R. Hamilton also indicated, and which consists in adopting the consideration that in the known values of the wave-lengths there is found to subsist very accurately the relation

$$l^2_F = \tfrac{1}{2}\,(l^2_H + l^2_B).$$

236. Availing himself of this circumstance, after reduction, and writing, for abridgement,

$$\frac{-\,(l^2_H - 2l^2_D + l^2_B)}{l^2_H - l^2_B} = A_D,$$

$$\frac{(l^2_D - l^2_B)}{(l^2_H - l^2_B)} A_D = B_D.$$

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237. The formula becomes (supposing instead of D any ray whose index is sought, and accordingly placing an accent instead of the letter D)

$$\mu_, - \mu_F = A_, (\mu_F - \mu_B) + B_, (\mu_H - 2\mu_F + \mu_B).$$

238. Here the coefficients $A_,$ and $B_,$, consisting wholly of quantities independent of the medium, are in the first instance determined from the wave-lengths, for each of the rays C, D, E, G, for which $\mu_,$ can thus be readily found, if the indices for B, F and H are assumed.

239. This method does not in its actual form give directly any of the constants in the series, but it will be easily seen that it enables us to determine the *limiting* value of μ, that is, (as we have seen) the constant p or H, by performing a similar calculation to that for any other index, but substituting the values of the constants (A) and (B) as obtained for a wave-length infinitely great compared with those of the other rays; that is in the formulas for A and B above, supposing the reciprocals of the square of the wave-length for the extreme ray in question to vanish, while those for the others remain finite.

240. Thus distinguishing these limiting values by subscribing 0, we find them to be,

$$A_0 = \frac{-(l^2{}_H + l^2{}_B)}{(l^2{}_H - l^2{}_B)},$$

$$B_0 = A_0 \frac{(-l^2{}_B)}{(l^2{}_H - l^2{}_B)}.$$

241. To take the most complete view of this method, however, we must return to the formula (234) which gives in general the means of calculating the in-

dices for C, D, E and G, those for B, F and H being assumed, apart from the introduction of the particular relation on which the preceding formulas are founded. Expressing it for brevity thus,

$$0 = (\mu_D - \mu_B)\,k - (\mu_F - \mu_B)\,l + (\mu_H - \mu_B)\,m.$$

The coefficients k, l, m, may readily be found from Fraunhofer's values of λ; and thus the results may be calculated.

242. For the corresponding relation of the other rays we must take a formula analogous to (234) which will be,

$$0 = \left\{ \begin{array}{l} \quad (\mu_C - \mu_B)\,(l^2{}_G - l^2{}_B)\,(l^2{}_E - l^2{}_B)\,(l^2{}_G - l^2{}_E) \\ -\,(\mu_E - \mu_B)\,(l^2{}_G - l^2{}_B)\,(l^2{}_C - l^2{}_B)\,(l^2{}_G - l^2{}_C) \\ +\,(\mu_G - \mu_B)\,(l^2{}_E - l^2{}_B)\,(l^2{}_C - l^2{}_B)\,(l^2{}_E - l^2{}_C), \end{array} \right.$$

which may be written as before,

$$0 = (\mu_C - \mu_B)\,k' - (\mu_E - \mu_B)\,l' + (\mu_G - \mu_B)\,m',$$

and k', l', m' found in the same manner. I gave some illustrations of this method in the *Journal of Science*, Vol. IX. p. 116.

REMARKS ON THE PRINCIPLE OF THE PRECEDING METHODS.

243. On comparing the several methods just considered it has already appeared that those last referred to, while more *exact* in their form, require larger assumptions from observation for their actual application. Under different forms the process in all of them is directly or indirectly equivalent to assuming *three* observed indices and interpolating the remaining four.

244. Now it may be said this actually carries us but a little way towards a real or satisfactory explanation, and that theory ought to assign a relation among the constants. In a word, it becomes a question of much interest and importance what amount of data it is fair to allow the theoretical computor to assume from observation, and to examine generally the legitimacy and sufficiency of such a solution of the problem.

245. It may perhaps be conceded that in the present state of our knowledge we must be content to regard the constants of the formula as unexplained by theory: yet even with this deficiency it may be contended that it is not an unimportant step to be able from three assumed values to assign the relation of the wave-length and refractive index, with a close approximation to the truth in all the intermediate cases.

246. But in order to come to any satisfactory opinion on these points, we must consider carefully what is the precise nature and object of the problem we have to solve. The question is in its nature rather one of philosophical logic than of mathematics, and presents some difficulty from the apparent absence of any analogous case in other branches of physical enquiry by which to guide our judgment. To this question I called attention in a short paper in the *Journal of Science*, (1839, Vol. XIV. p. 261,) and at that time had come to a conclusion *against* the legitimacy of the process requiring the assumption of three indices, and contended for such methods alone as involved only *two* constants.

247. The argument by which I was led to this opinion was derived from the manifest independence of the two questions of *refraction* and *dispersion;* in accordance

with which it would seem reasonable that one datum for each medium should be a *refraction* constant and one more a *dispersion constant.* The former would consist in the limiting index, while the other would be found in the value of the arc θ. This is what is in fact exhibited by the approximate formula before mentioned, a formula too, which is unquestionably very nearly accordant with the truth through a considerable range of actual instances. It becomes, then, important to consider whether any *theoretical* conditions are conceivable which may warrant the adoption of it; and this reduces itself to the consideration, whether in the summation of different combinations of $H_{\prime}$ and $\Delta x_{\prime}$, of $H_{\prime\prime}$ and $\Delta x_{\prime\prime}$, &c. in the form above exhibited, we may not discover some ground for justifying the adoption of a *single term* containing some *mean value* of Δx combined with a constant factor H', such that we may have accurately or nearly, as before explained,

$$\Sigma \left\{ H \frac{\sin \theta}{\theta} \right\} = H' \left(\frac{\sin \theta}{\theta} \right)$$

248. How far the conditions of a system of molecules, such as that supposed in the theory before developed, may be found susceptible of leading to such a deduction, must be left for examination on the grounds of dynamical analysis.

249. But at all events, for those who may concur in this view, it will be necessary to compute by some such method as that before pointed out, (223) all the indices which have not as yet been compared with the theory: viz. those given in Table III. of my Report to the British Association, as well as to recalculate those before computed by a merely tentative process.

250. Yet in coming to the result which I expressed in the paper just referred to, I was still anxious to find any sound vindication of the previously received process, as being a mode of proceeding which possesses many recommendations, and has afforded some valuable numerical accordances.

251. In any point of view it is essential to the settlement of the question, either to make out a vindication of the sufficiency of mere methods of interpolation, or to find some successful method of computing from *two* observed indices and to apply it throughout all the cases required.

To pursue *one* of these courses is clearly the only alternative for completing the investigation: one or the other must be followed out if the undulating theory is to be effectually relieved from the stigma of leaving the dispersion of light still unexplained.

252. On a careful reconsideration of the whole subject, however, I have been led to doubt the correctness of my criticism: and in order to state more clearly the grounds for a revision of my former opinion, I will pursue the case as partially stated in the paper last referred to: I there observed,—" To all who have attentively considered the nature of those remarkable optical properties of bodies which are called their *refractive* and *dispersive powers*, it will be evident that we have *a very peculiar* case to consider. The problem we have to solve is rather *a combination of two distinct problems.* The dispersive and refractive powers follow no proportion to each other, and it is almost impossible to conceive any theory, or even any empirical mathematical law, which could connect the two together. For example, we have diamond

and water, with refractions nearly double the one of the other, and dispersions nearly the same. Flint glass and oil of cassia with the same refractions nearly, and dispersions as about 1 to 3. In a word, the absolute magnitude of the deviation of any given ray, or of white light, bears no relation whatever to the difference of deviation between the extreme rays of the spectrum, in different media. If then we seek a theory to explain the facts, it would be not only unreasonable to expect it to connect such obviously incongruous phænomena, but it ought most rationally to involve *two distinct constants*, one belonging to the *refractive* POWER, the other to the *refractive* CHARACTER of the medium. And in conformity with the general conditions of formulas of this kind, we might expect that, directly or indirectly, we should have to assume *two* quantities as given by observation, in any calculation to compare theory with observation, for the spectrum of a particular medium."

253. But this second problem, again considered in itself, is not a simple one; it involves two considerations, at first sight at least, quite distinct: viz. first, the total *amount* of the dispersion or expansion of the spectrum; and second, the relative expansions of particular coloured spaces: that is, the relative deviations of the two *extreme* rays, and those of *intermediate* rays. Or, in other words, as we have spoken of *two* points, the refractive *power* and the refractive *character* of the medium, so under this second head we have again to consider the *dispersive power*, and the *dispersive character*.

254. Or we may illustrate it thus:—The total lengths of spectra in two media are in a ratio R, the intervals between two given rays are in another ratio R', the absolute deviations of the same ray, in a third ratio R''.

We have thus three different classes of relations to investigate, and the general problem depends on the three jointly.

255. Or thus more generally :—The theory ought to include, first, the law of the variation of the indices from one ray to another in the same medium as a function of the wave-length, which we will call $F(\lambda)$.

Secondly, the law of variation of the indices of the same ray from one medium to another, which may depend on the former, and which may be called $F'\{F(\lambda)\}$.

Thirdly, The final resulting law of the absolute magnitudes of the indices, again dependent on both the foregoing, and which we may call $F''[F'\{F(\lambda)\}]$.

Hence it would appear that *each* of these three laws of variation demands *one* independent constant.

256. If, then, this illustration be admitted, it justifies the forms in which the theory has been already developed, or makes the absolute position of a given ray depend on three constants, which must directly or indirectly be derived from three observed indices.

Still I am most desirous that the question should be further enquired into, and regard it as still fairly open to difference of opinion.

257. One of the constants (in any view of the theory) is, as we have seen, that remarkable quantity the *limit of all refraction:* this points to an entirely new property of refractive media, which I believe was never before so much as suspected. No means have as yet appeared for verifying it experimentally, unless by a reference to the

refraction of heat: and this, in the present state of our knowledge, it would seem possible to study only in the case of that singular substance rock salt.

REMARKS ON FORMER CALCULATED RESULTS.

258. As far as Fraunhofer's series of indices are concerned, the close accordances of the results as calculated by my first *approximate* process, have been abundantly confirmed by the *exact* methods of Mr Kelland and M. Cauchy. Those, however, who may concur in the views just now adverted to, as to the inadmissibility of the assumption of so many as three indices, will of course feel it desirable to have the calculations repeated by the *direct* method above described assuming only two indices from observation. But there can be no doubt that such a process will succeed as well in these cases as my original indirect method did.

259. With regard to my second series, or Rudberg's indices, as they were computed likewise by the same tentative method, and no one, I believe, has attempted any other calculation of them, it would manifestly be most material to have them recomputed by a more exact method.

260. The third series consists of cases taken from my first approximate observations of indices, (including those of the more important media) calculated by Sir W. R. Hamilton's method of interpolation. Now even if the principle of that method be allowed to be sufficient, this series demands an entirely fresh computation; since those data have been superseded by the more accurate values, furnished by the later observations, as given in my Report on Refractive Indices to the British Association.

261. By the same method, the indices for the rays D and G in all Fraunhofer's media were calculated by Sir W. R. Hamilton himself, and the results were given in my papers in the *Journal of Science.* In these results there appeared some small deviations from the observed values. Now (as we have seen) the constants A and B in this computation are derived solely from the values of λ for each ray taken from the well known determinations of Fraunhofer from interference. It is doubtless, possible, that these determinations may have been affected by errors. Sir W. R. Hamilton undertook to calculate what amount of alteration in the values of λ would remove those discrepancies.

262. In my series No. III, the extreme cases, as oil of cassia, &c., exhibited considerable discrepancies. I repeated a similar calculation: but found that to remove them a much larger alteration than could be for a moment allowed in Fraunhofer's data, must be supposed in order to produce any sensible effect.

263. The object in the series No. IV. was to try whether the method of Mr Kelland would succeed better in the cases last alluded to. It does so to some degree in oil of cassia; but in the two cases of sulphuret of carbon there are still considerable deviations. There is in each case the same discrepancy in the ray G which the author notices in his own results.

The data, however, it must be remembered, are only those furnished from approximate observations. New computations, therefore, are requisite for the more accurate values since obtained.

264. Mr Kelland has (in his paper in the *Camb. Trans.*) given the values of p for all Fraunhofer's media:

the square root of whose reciprocal is the limiting index, which is also the same as the empirical constant adopted in my first tentative method. For the sake of comparison I will here give the mean values taken from Mr Kelland's results, nearly, along with the constants used in my researches No. I.

Medium.	Limiting index $\frac{1}{\sqrt{p}} = \frac{1}{H}$.	
	Exact Method.	Approximate Method.
Flint glass... 13	1.6090	1.6070
.... 23	1.6089	1.6060
.... 30	1.6069	1.6033
.... 3	1.5860	1.5820
Crown glass . M	1.5438	1.5430
.... 9	1.5158	1.5162
.... 13	1.5147	1.5145
Oil Turpentine..	1.4596	1.4590
Sol. of Potass...	1.3910	1.3922
Water...........	1.3236	1.3243

265. The direct calculation assuming only two indices, may be easily made by the formula before given (223) by those who may adopt that view of the subject. I will only add, that when impressed with the belief of its importance, I made trial of that method for the extreme case of oil of cassia (*Report*, Tab. III. No. 1.) that is, assuming *any two* indices, a value of θ is found, and thence H. If the supposition were exact, these values ought to result equal as deduced from every pair of rays. But, in fact, in this way values of θ successively derived from the rays B and C, from B and D, from B

and E, &c., are found to differ, so as to give the following values for the constant:

Calculated from Rays.	Values of $\frac{1}{H}$.	Difference from Mean.
B, C	1.5625	+ .0029
B, D	1.5620	+ .0024
B, E	1.5677	+ .0081
B, F	1.5583	− .0013
B, G	1.5546	− .0050
B, H	1.5528	− .0068
Mean	1.5596	

266. These differences are of course too great to be allowed. It remains then with those who may adhere to this view of the subject to devise some improvement upon the formula: which perhaps may not be an unpromising object of research, when we look at the circumstance that the values derived from the lower rays are all in excess, those from the higher in defect.

267. If the principle of the three indices be admitted, Sir W. R. Hamilton's method possesses pre-eminent recommendations in point of facility and brevity in the numerical computations. As I have, at least for the present, been led to use it extensively, and as it may be employed by others, it may be worth while here to give some details which are useful in the practical application.

268. The determination of the constants A and B for each of the rays to be found, is (as we have seen) independent of the medium, and is therefore made once for all from the values of λ. The application of the

formulas (236) requires the use of the following table, which may be also recurred to for verifying the numbers given.

269. In order to avoid cyphers, supposing Fraunhofer's values of λ multiplied by 1000, we have as follows:

Ray.	λ.	$\frac{1}{\lambda^2}$.
B	.2541	15.488
C	.2422	17.047
D	.2175	21.139
E	.1945	26.434
F	.1794	31.071
G	.1587	39.705
H	.1464	46.658

270. Hence by the formula (236) we obtain

$$\log(-A_C) = \bar{1}.95433, \quad \log(-B_C) = \bar{2}.65253,$$

$$\log(-A_D) = \bar{1}.80441, \quad \log(-B_D) = \bar{1}.06281,$$

$$\log(-A_E) = \bar{1}.49646, \quad \log(-B_E) = \bar{1}.03196,$$

$$\log(-A_G) = \bar{1}.74341, \quad \log(-B_G) = \bar{1}.63380.$$

Also for the limit, we have by (240)

$$\log(-A_0) = 0.29968, \qquad \log(B_0) = \bar{1}.99594.$$

271. From the above table the remark (235) is verified, for

$$\tfrac{1}{2}(l_B^2 + l_H^2) = 31.073 = l_F^2, \text{ very nearly.}$$

272. Should it be found desirable to calculate by the more general formulas (241) (242) the following values of the coefficients will be useful: they result from the values in the table (269) nearly as follows:

$$\log k = 3.87943, \qquad \log k' = 3.54623,$$

$$\log l = 3.65263, \qquad \log l' = 3.93137,$$

$$\log m = 3.94185, \qquad \log m' = 2.28794.$$

SECTION VII.

CALCULATED RESULTS.

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273. In the ensuing articles I shall exhibit the actual results of calculation by theory, for *all the indices for definite rays hitherto determined* by observation. The computations are all made on a uniform plan by Sir W. R. Hamilton's formula. For convenience I have arranged them in six sets. The two first containing the indices of Fraunhofer and Rudberg, the remaining four those observed by myself, and adopted exactly as given in my Report to the British Association, to which the reader is referred for all particulars relative to the determination of them.

I.

274. The indices of Fraunhofer are here computed for the sake of uniformity and comparison by the same formula. The rays *D* and *G* were calculated by the author of the formula, as before stated. The rest have been supplied by myself; but they have no pretensions to being followed out to any great numerical accuracy. The addition of the value of the limiting index in every case will assist in comparing this method with that of Mr Kelland. These limiting values may also be compared with the same quantities as adopted empirically in my approximate calculations in the first series.

275. Flint Glass. No. 13.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6095	
B	1.6277	....	
C	1.6297	1.6297	.0000
D	1.6350	1.6349	− .0001
E	1.6420	1.6416	− .0004
F	1.6483	....	
G	1.6603	1.6606	+ .0003
H	1.6711	....	

276. Flint Glass. No. 23.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6093	
B	1.6256	....	
C	1.6285	1.6285	.0000
D	1.6337	1.6335	− .0002
E	1.6405	1.6402	− .0003
F	1.6468	....	
G	1.6588	1.6591	+ .0003
H	1.6697	....	

277. Flint Glass. No. 30.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6060	
B	1.6236	....	
C	1.6255	1.6254	− .0001
D	1.6306	1.6305	− .0001
E	1.6373	1.6371	− .0002
F	1.6435	....	
G	1.6554	1.6557	+ .0003
H	1.6661	....	

278. Flint Glass. No. 23.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5866	
B	1.6020	....	
C	1.6038	1.6037	− .0001
D	1.6085	1.6082	− .0003
E	1.6145	1.6142	− .0003
F	1.6200	....	
G	1.6308	1.6310	+ .0002
H	1.6404	....	

279. Crown Glass. M.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5438	
B	1.5548	....	
C	1.5559	1.5560	+ .0001
D	1.5591	1.5590	− .0001
E	1.5631	1.5630	− .0001
F	1.5667	....	
G	1.5735	1.5737	+ .0002
H	1.5795	....	

280. Crown Glass. No. 30.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5147	
B	1.5243	....	
C	1.5253	1.5253	.0000
D	1.5280	1.5279	− .0001
E	1.5314	1.5312	− .0002
F	1.5343	....	
G	1.5399	1.5400	+ .0001
H	1.5447	....	

281. Crown Glass. No. 9.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5159	
B	1.5258	....	
C	1.5268	1.5268	.0000
D	1.5296	1.5294	– .0002
E	1.5330	1.5328	– .0002
F	1.5360	....	
G	1.5416	1.5418	+ .0002
H	1.5465	....	

282. Oil of Turpentine, temp. $10^0.6$.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.4600	
B	1.4705	....	
C	1.4715	1.4715	.0000
D	1.4744	1.4744	.0000
E	1.4783	1.4782	– .0001
F	1.4817	....	
G	1.4882	1.4883	+ .0001
H	1.4939	....	

283. Solution of Potass, temp. $21^0.25$.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3913	
B	1.3996	....	
C	1.4005	1.4005	.0000
D	1.4028	1.4027	– .0001
E	1.4056	1.4054	– .0002
F	1.4080	....	
G	1.4126	1.4127	+ .0001
H	1.4164	....	

284. Water, temp. $18^0.75$.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3237	
B	1.3309	....	
C	1.3317	1.3316	− .0001
D	1.3336	1.3334	− .0002
E	1.3358	1.3357	− .0001
F	1.3378	....	
G	1.3413	1.3414	+ .0001
H	1.3442	....	

II.

285. The results for the media examined by Rudberg are here for the first time calculated by an exact and direct method. It will be satisfactory to compare them with those of the indirect method in my *Researches*, No. II., especially with regard to the limit or constant; as also with the similar results obtained by Mr Kelland's method, should any one think it worth the labour of re-computing them.

286. Calc Spar. Ordinary Ray.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6386	
B	1.6531	....	
C	1.6545	1.6546	+ .0001
D	1.6585	1.6585	.0000
E	1.6636	1.6633	− .0003
F	1.6680	....	
G	1.6762	1.6763	+ .0001
H	1.6833	....	

287 Calc Spar. Extraordinary Ray.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.4774	
B	1.4839	....	
C	1.4845	1.4846	+ .0001
D	1.4863	1.4864	+ .0001
E	1.4887	1.4886	− .0001
F	1.4907	....	
G	1.4945	1.4944	− .0001
H	1.4978	....	

288. Aragonite. First Axis.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5204..	
B	1.5275..	....	
C	1.5282	1.5283	+ .0001
D	1.5301	1.5302	+ .0001
E	1.5326	1.5325	− .0001
F	1.5348..	....	
G	1.5388	1.5388	.0000
H	1.5422..	....	

289. Aragonite. Third Axis.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6625..	
B	1.6763..	....	
C	1.6778	1.6778	.0000
D	1.6816	1.6815	− .0001
E	1.6863	1.6861	− .0002
F	1.6905..	....	
G	1.6984	1.6983	− .0001
H	1.7051..	....	

290. Aragonite. Second Axis.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6666..	
B	1.6806..	....	
C	1.6820	1.6821	+ .0001
D	1.6859	1.6859	.0000
E	1.6908	1.6906	− .0002
F	1.6951..	....	
G	1.7032	1.7032	.0000
H	1.7101..	....	

291. Quartz. Extraordinary Ray.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5411	
B	1.5499	....	
C	1.5508	1.5508	.0000
D	1.5533	1.5532	− .0001
E	1.5563	1.5561	− .0002
F	1.5589	....	
G	1.5636	1.5638	+ .0002
H	1.5677	....	

292. Quartz. Ordinary Ray.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5322	
B	1.5409	....	
C	1.5418	1.5418	.0000
D	1.5442	1.5441	− .0001
E	1.5471	1.5470	− .0001
F	1.5496	....	
G	1.5542	1.5544	+ .0002
H	1.5582	....	

293. Topaz. First Axis.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6095	
B	1.6179	....	
C	1.6188	1.6189	+ .0001
D	1.6211	1.6211	.0000
E	1.6241	1.6239	− .0002
F	1.6265	....	
G	1.6312	1.6312	.0000
H	1.6351	....	

294. Topaz. Second Axis.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6016	
B	1.6084	....	
C	1.6093	1.6093	.0000
D	1.6116	1.6117	+ .0001
E	1.6145	1.6144	− .0001
F	1.6170	....	
G	1.6215	1.6213	− .0002
H	1.6254	....	

295. Topaz. Third Axis.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5978	
B	1.6105	....	
C	1.6114	1.6114	.0000
D	1.6137	1.6136	− .0001
E	1.6167	1.6164	− .0003
F	1.6191	....	
G	1.6236	1.6238	+ .0002
H	1.6274	....	

III.

296. Of my own indices I will first give those which belong to certain solutions of low dispersive powers and little importance, and for which the calculation was in consequence only carried to a slight degree of approximation.

297. Solution of Muriate of Ammonia.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3432	
B	1.3499	....	
C	1.3508	1.3511	+ .0003
D	1.3529	1.3528	− .0001
E	1.3552	1.3553	+ .0001
F	1.3575	....	
G	1.3617	1.3614	− .0003
H	1.3650	....	

298. Solution of Nitrate of Potass.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3382	
B	1.3457	....	
C	1.3468	1.3465	− .0003
D	1.3487	1.3485	− .0002
E	1.3510	1.3510	.0000
F	1.3533	....	
G	1.3586	1.3575	− .0011
H	1.3608	....	

299. Solution of Sulphate of Magnesia.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3365	
B	1.3434	....	
C	1.3442	1.3441	– .0001
D	1.3462	1.3460	– .0002
E	1.3486	1.3483	– .0003
F	1.3504	....	
G	1.3540	1.3542	+ .0002
H	1.3570	....	

300. Solution of Nitrate of Mercury.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3330	
B	1.3408	....	
C	1.3419	1.3416	– .0003
D	1.3439	1.3437	– .0002
E	1.3462	1.3463	+ .0001
F	1.3487	....	
G	1.3528	1.3530	+ .0002
H	1.3560	....	

301. Solution of Muriate of Barytes.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3331	
B	1.3398	....	
C	1.3406	1.3405	– .0001
D	1.3421	1.3423	+ .0002
E	1.3438	1.3445	– .0007
F	1.3466	....	
G	1.3504	1.3503	– .0001
H	1.3531	....	

302. Solution of Sulphate of Soda.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3319	
B	1.3392	....	
C	1.3398	1.3398	.0000
D	1.3419	1.3417	− .0002
E	1.3442	1.3440	− .0002
F	1.3462	....	
G	1.3499	1.3501	+ .0002
H	1.3528	....	

303. Solution of Nitrate of Bismuth.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3231	
B	1.3306	....	
C	1.3315	1.3310	− .0005
D	1.3332	1.3329	− .0003
E	1.3355	1.3352	− .0003
F	1.3374	....	
G	1.3410	1.3413	+ .0003
H	1.3437	....	

304. Solution of Nitrate of Lead.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3385	
B	1.3455	....	
C	1.3461	1.3464	+ .0003
D	1.3482	1.3483	+ .0001
E	1.3506	1.3506	.0000
F	1.3528	....	
G	1.3568	1.3567	− .0001
H	1.3600	....	

305. Solution of Superacetate of Lead.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3355	
B	1.3429	....	
C	1.3437	1.3434	−.0003
D	1.3455	1.3453	−.0002
E	1.3480	1.3476	−.0004
F	1.3498	....	
G	1.3538	1.3537	−.0001
H	1.3571	....	

306. Solution of Subacetate of Lead.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3274	
B	1.3350	....	
C	1.3357	1.3353	−.0004
D	1.3373	1.3372	−.0001
E	1.3398	1.3395	−.0003
F	1.3417	....	
G	1.3453	1.3456	+.0003
H	1.3481	....	

IV.

307. In my *Report*, p. 12, it will be seen that there are inserted results for certain media, which are specially distinguished as being only rough approximations; since peculiarities in those media seem to preclude the hope of obtaining accurate measures. Some of these were included in the calculations of my *Researches*, No. III. Though they are perhaps hardly worthy of being made subjects of computation, I have yet thought it desirable to complete the series: but of course no great value can be attached to them.

308. Oil of Angelica.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.471	
B	1.484	....	
C	1.486	1.485	− .001
D	1.489	1.488	− .001
E	1.493	1.492	− .001
F	1.496	....	
G	1.505	1.503	− .002
H	1.509	....	

309. Liquid Ammonia.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.339	
B	1.345	....	
C	1.346	1.347	+ .001
D	1.348	1.349	+ .001
E	1.350	1.351	+ .001
F	1.353	....	....
G	1.355	1.357	+ .002
H	1.360	....	

310. Chromate of Lead.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.365	
B	1.369	....	
C	1.372	1.373	+ .001
D	1.374	1.375	+ .001
E	1.376	1.377	+ .001
F	1.379	....	
G	1.384	1.383	− .001
H	1.389	....	

311. Chromate of Potass.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.346	
B	1.351	....	
C	1.352	1.354	+ .002
D	1.353	1.356	+ .003
E	1.357	1.358	+ .001
F	1.360	....	
G	1.364	1.364	.000
H	....	....	

312. Oil of Cummin.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.487	
B	1.502	....	
C	1.504	1.503	− .001
D	1.507	1.507	.000
E	1.513	1.513	.000
F	1.520	....	
G	1.532	1.531	− .001
H	1.543	....	

313. Oil of Pimento.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.518	
B	1.528	....	
C	1.532	1.530	− .002
D	1.535	1.535	.000
E	1.542	1.541	− .001
F	1.549	....	
G	1.559	1.560	+ .001
H	1.571	....	

314. Balsam of Peru.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.561	
B	1.585	....	
C	1.587	1.587	.000
D	1.593	1.593	.000
E	1.603	1.603	.000
F	1.613	....	
G	1.634	1.633	− .001
H	1.653	....	

V.

315. My next series comprises some media higher in the scale than those in the first, and presents some cases of more interest. Here, as in the remaining instances, the computation was conducted with more accuracy, though in all cases it might have been carried further had the circumstances appeared to render it desirable.

It will be remarked that the last case, that of Muriate of Zinc, offers a singular discrepancy in the rays *C* and *D*. I am unable to determine whether this must be regarded as a real unexplained anomaly, or whether it may be attributed to some error in making the observation or deducing the index.

316. Sulphuric Acid, temp. 18°.6.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.4232	
B	1.4321	....	
C	1.4329	1.4329	.0000
D	1.4351	1.4351	.0000
E	1.4380	1.4377	– .0003
F	1.4400	....	
G	1.4440	1.4437	– .0003
H	1.4463	....	

317. Muriatic Acid. temp. $18^0.6$.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3853	
B	1.4050	....	
C	1.4060	1.4062	+ .0002
D	1.4095	1.4090	– .0005
E	1.4130	1.4126	– .0004
F	1.4160	....	
G	1.4217	1.4217	.0000
H	1.4261	....	

318. Nitric Acid, temp. $18^0.6$

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3893	
B	1.3988	....	
C	1.3998	1.3999	+ .0001
D	1.4026	1.4025	– .0001
E	1.4062	1.4060	– .0002
F	1.4092	....	
G	1.4155	1.4153	– .0002
H	1.4206	....	

319. Alcohol, temp. $17^0.6$.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3532	
B	1.3628	....	
C	1.3633	1.3635	+ .0002
D	1.3654	1.3656	+ .0002
E	1.3675	1.3675	.0000
F	1.3696	....	
G	1.3733	1.3732	– .0001
H	1.3761	....	

320. Pyrolignous Acid.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3709	
B	1.3729	....	
C	1.3745	1.3738	− .0007
D	1.3760	1.3758	− .0002
E	1.3785	1.3783	− .0002
F	1.3807	....	
G	1.3848	1.3849	+ .0001
H	1.3884	....	

321. Solution of Soda. 16°.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3929	
B	1.4036	....	
C	1.4039	1.4046	+ .0007
D	1.4075	1.4072	− .0003
E	1.4109	1.4105	− .0004
F	1.4134	....	
G	1.4181	1.4187	+ .0006
H	1.4221	....	

322. Solution of Potass. 16°.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3942	
B	1.4024	....	
C	1.4036	1.4034	− .0002
D	1.4061	1.4057	− .0004
E	1.4091	1.4082	− .0009
F	1.4117	....	
G	1.4162	1.4172	+ .0010
H	1.4199	....	

323. Water. 15°.8.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3246	
B	1.3317	....	
C	1.3326	1.3324	− .0002
D	1.3343	1.3343	.0000
E	1.3364	1.3365	+ .0001
F	1.3386	....	
G	1.3429	1.3424	− .0005
H	1.3448	....	

324. Solution of Muriate of Lime.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3953	
B	1.4006	....	
C	1.4016	1.4016	.0000
D	1.4040	1.4040	.0000
E	1.4070	1.4070	.0000
F	1.4099	....	
G	1.4150	1.4150	.0000
H	1.4190	....	

325. Solution of Muriate of Zinc.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.3283	
B	1.3351	....	
C	1.3402	1.3366	− .0036
D	1.3421	1.3398	− .0023
E	1.3444	1.3435	− .0009
F	1.3466	....	
G	1.3501	1.3509	+ .0008
H	1.3534	....	

VI.

326. The sixth series contains all the most interesting and important cases, including those of the highest dispersive power yet examined, and in which the theory is obviously put to a more exact test.

327. Oil of Sassafras, temp. 17°.2.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5120	
B	1.5257	....	
C	1.5275	1.5274	–.0001
D	1.5321	1.5320	–.0001
E	1.5387	1.5384	–.0003
F	1.5448	....	
G	1.5575	1.5576	+.0001
H	1.5693	....	

328. Kreosote, temp. 18°.2.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5164	
B	1.5320	....	
C	1.5335	1.5339	+.0004
D	1.5383	1.5387	+.0004
E	1.5452	1.5450	–.0002
F	1.5515	....	
G	1.5639	1.5639	.0000
H	1.5749	....	

329. Rock Salt.

Ray.	Index observed.	Index calculated.	
Limit	....	1.5277	
B	1.5403	....	
C	1.5415	1.5417	+ .0002
D	1.5448	1.5452	+ .0004
E	1.5498	1.5497	− .0001
F	1.5541	....	
G	1.5622	1.5622	.0000
H	1.5691	....	

330. Oil of Anise, temp. 20°.9. (No. vii. Report.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5286	
B	1.5451	....	
C	1.5473	1.5472	− .0001
D	1.5534	1.5534	.0000
E	1.5623	1.5617	− .0006
F	1.5707	....	
G	1.5881	1.5886	+ .0005
H	1.6053	....	

331. Oil of Anise, temp. 13°.25. (No. vi.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5315	
B	1.5482	....	
C	1.5504	1.5504	.0000
D	1.5565	1.5564	− .0001
E	1.5650	1.5646	− .0004
F	1.5733	....	
G	1.5901	1.5907	+ .0006
H	1.6066	....	

332. Oil of Anise, temp. $15^{0}.1$. (No. v.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5312	
B	1.5486	....	
C	1.5508	1.5508	.0000
D	1.5572	1.5570	− .0002
E	1.5659	1.5653	− .0006
F	1.5743	....	
G	1.5912	1.5922	+ .0010
H	1.6084	....	

333. Sulphuret of Carbon, temp. $15^{0}.65$.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.6002	
B	1.6182	....	
C	1.6219	1.6211	− .0008
D	1.6308	1.6291	− .0017
E	1.6438	1.6418	− .0020
F	1.6555	....	
G	1.6799	1.6843	+ .0044
H	1.7020	....	

334. Oil of Cassia, temp. $22^{0}.5$. (No. iii.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5728	
B	1.5895	....	
C	1.5930	1.5926	+ .0004
D	1.6026	1.6022	− .0004
E	1.6174	1.6156	− .0018
F	1.6314	....	
G	1.6625	1.6654	+ .0029
H	1.6985	....	

335. Oil of Cassia, temp. 14°. (No. ii.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5777	
B	1.5945	....	
C	1.5979	1.5975	− .0004
D	1.6073	1.6066	− .0007
E	1.6207	1.6202	− .0005
F	1.6358	....	
G	1.6671	1.6695	+ .0024
H	1.7025	....	

336. Oil of Cassia, temp. 10°. (No. i.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5762	
B	1.5963	....	
C	1.6007	1.5996	− .0011
D	1.6104	1.6092	− .0012
E	1.6249	1.6232	− .0017
F	1.6389	....	
G	1.6698	1.6721	+ .0023
H	1.7039	....	

337. The four highest cases, viz. Oil of Cassia (Nos. i., ii., and iii.) and Sulphuret of Carbon, as here calculated, give differences much too large to be allowed. It remains to be considered in what respect the theory can be further modified, or extended, so as to afford any hope of computing them with better success.

338. As a first step, it occurred to me to examine whether by empirically altering the constant coefficients, for each medium, better accordances might be obtained. I have tried this only in the coefficient B; and find that when that coefficient is multiplied by a new factor C, constant for the medium, accordances are obtained which I consider to be quite within the limits of errors of observation. For the sake of comparison, I have extended this correction also to the case of Oil of Anise (No. v.). But the differences in the other instances are not so great as to render it desirable to go through similar calculations for them.

339. I find for the values of C in the several cases,

Oil of Cassia, No. i. ...	log C	$= \bar{1}.840.$
———— No. ii. ...		$= \bar{1}.900.$
———— No. iii. ...		$= \bar{1}.878.$
Sulphuret of Carbon ...		$= \bar{1}.679.$
Oil of Anise, No. v. ...		$= \bar{1}.860.$

With this correction, the results are as follows:

340. Oil of Cassia. (No. i.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5693	
B	1.5963	....	
C	1.6007	1.5999	−.0008
D	1.6104	1.6101	−.0003
E	1.6249	1.6240	−.0009
F	1.6389	....	
G	1.6698	1.6691	−.0007
H	1.7039	....	

341. Oil of Cassia. (No. ii.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5726	
B	1.5945	....	
C	1.5979	1.5970	−.0009
D	1.6073	1.6072	−.0001
E	1.6207	1.6207	.0000
F	1.6358	....	
G	1.6671	1.6673	+.0002
H	1.7025	....	

342. Oil of Cassia. (No. iii.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5667	
B	1.5895	....	
C	1.5930	1.5929	−.0001
D	1.6026	1.6028	+.0002
E	1.6174	1.6163	−.0011
F	1.6314	....	
G	1.6625	1.6627	+.0002
H	1.6985	....	

343. Sulphuret of Carbon.

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5721	
B	1.6182	....	
C	1.6219	1.6216	− .0003
D	1.6308	1.6307	− .0001
E	1.6438	1.6429	− .0009
F	1.6555	....	
G	1.6799	1.6800	+ .0001
H	1.7020	....	

344. Oil of Anise. (No. v.)

Ray.	Index observed.	Index calculated.	Difference.
Limit	....	1.5293	
B	1.5486	....	
C	1.5508	1.5510	+ .0002
D	1.5572	1.5573	+ .0001
E	1.5659	1.5657	− .0002
F	1.5743	....	
G	1.5912	1.5912	.0000
H	1.6084	....	

REFRACTION OF HEAT IN ROCK SALT.

345. The case of Rock Salt demands a further remark, on account of its remarkable relations to heat.

In the third part of his Researches on Heat, &c. (*Edin. Trans.* XIV.) Prof. Forbes gives the indices of refraction resulting from his observations for several kinds of heat from different sources, and compares them with the mean index for light in this medium. But he observes that these results, though sufficient for the purpose of a *relative* estimate, are by no means *absolutely* exact. Comparing then the apparent mean index for light as obtained by this method, with that derived from a more direct and exact process, he considers that a correction of .04 *or* .05 should be applied to all the indices before given, (p. 29). The indices for heat of *all* kinds (if the theory be true) ought to fall *within the limit* assigned by the above calculation.

346. Prof. Forbes determines the index for mean light by the same method as those for heat. Now he describes two observations; one with a very small luminous source, the other with the flame of the Locatelli lamp, the same as used in the heat observations: these results differ, and the author takes the mean. He however observes that the latter is the more fair mode of *comparison* with the heat observations: and this I think so manifestly just that this latter result *alone* appears to me preferable to the mean, where the object is *comparative* results. Now this gives, in the author's notation, $ab = 15.75$, whence there results $\mu = 1.598$. And since the author

allows a latitude as to the correction, we may adopt $-.04$ as its value. This gives for mean light the corrected index $\mu = 1.558$.

347. Of the different species of heat examined by Prof. Forbes, several groups may be formed, consisting of those whose indices are nearly equal. They are thus classified in the following table, with the mean index for each group as approximately found, as well as that for white light, and as corrected by the number proposed.

For comparison I have added the mean of the extreme indices, viz. those of B and H, and that for B, as observed above, and the calculated value of the limit.

348. On other grounds, Prof. Forbes has determined the mean length of a wave of dark heat to be $\lambda = .000079$ inch. For such a ray, we may calculate as in the other cases the values of A and B, and thus obtain the theoretical index.

Thus I find for such a ray

$$\log(-A) = 0.27668, \qquad \log(B) = \bar{1}.92552.$$

Whence by Sir W. R. Hamilton's formula

$$\mu = 1.529.$$

349. The following table will present all these results in one point of view:

ROCK SALT.

Rays.	Indices (Forbes).		Indices as above.
	Approximate.	Corrected by $-.04$.	
White light (Locatelli lamp	1.598	1.558	
Mean of B and H......	....	....	1.5547
——— B	....	....	1.5403
Heat from			
Locatelli lamp, with alum .... — opake glass......	1.595	1.555	
— window glass ... — mica .... Incandesent platina, with glass ...... — opake mica......	1.586	1.546	
Locatelli lamp direct .... Incandesent platina . Mercury at 450°..... Dark hot brass, with mica	1.573	1.533	
Dark hot brass	1.568	1.528	
Ray $\lambda = .000079$	....	....	1.529
Limit	....	....	1.5277

350. Upon the whole, considering that Prof. Forbes expressly states the results to be open to some uncertainty, I think the accordance here exhibited between each class of observations, and of both with theory is as

close as the degree of accuracy of either set of results should lead us to expect.

INFLUENCE OF TEMPERATURE.

351. In such of the above cases, (viz. Nos. 283, 284; 322, 323; 330, 331, 332, and 340, 341, 342,) as exhibit results for the same medium at different temperatures, it is obvious, that if the theoretical formula be allowed to give a sufficient accordance in each case, it virtually includes *an explanation of the influence of temperature.* But it is equally true that the theoretical ground of such an explanation is not explicitly indicated; and the principles, as well as details, of such an application remain to be investigated.

CONCLUSION.

352. From the degree of accordance thus exhibited, in so large a range of instances, between observation and theory, we have to form our judgment to what extent it can be regarded as affording a satisfactory approach towards the expression and explanation of the actual law of nature. And, upon the whole, I conceive it is distinctly shewn that the formula, as strictly deduced from theory, applies sufficiently well to the cases of all media whose dispersion does not exceed that of oil of anise. It also represents, with a certain general approximation to truth, the indices of some more highly dispersive bodies. It is therefore extremely probable that the essential principle of the theory has some real foundation in nature. While looking at the regularity of the deviation, and the readiness with which the mere introduction of a new constant removes it, it seems likely that the theory only requires some slight further development, or extension, to warrant the introduction of such a factor, in order to make it apply accurately to the higher cases, while it still includes the simpler form which so well accords with the lower.

353. Not being able myself satisfactorily to investigate any grounds for such a modification, and impressed with the necessity of not assuming more data from observation, I have only to express a hope that these remarks may give occasion to a further examination of the subject from some of those who are so able to grapple with its difficulties; and that they may either succeed in discovering

theoretical grounds for such an improvement in the formula as is here hypothetically adopted, or devise altogether some more complete view of dispersion.

354. It must however be borne in mind, also, that at present our knowledge of refractive indices is confined within narrow limits: and there are numerous cases to which it has not as yet been found practicable to apply prismatic examination, and among them some of the most interesting, from their known high position in the scale; such as realgar, sulphur, and chromate of lead. When such cases shall have been successfully subjected to prismatic observation, it is probable the theory will be put to a far more severe test than it has yet been.

355. But with so strong a presumption as we already have in its favour, the most reasonable course in the meanwhile is, indisputably, to review carefully its first principles, and examine into whatever further development or modification it may be capable of receiving, with the view to meet any such further demands as may be made upon its resources.

ADDENDUM.

It may be convenient to the Student to add the following illustrations of certain points in the deductions connected with the wave surface, (191).

The expressions for the directions of single-ray-velocity and single-normal-velocity there given are thus obtained.

The first result directly from the expressions (187), since

$$\frac{z}{b} = \cos \rho' z, \quad \text{and } \frac{x}{b} = \cos \rho' x.$$

The second expressions are deduced thus:

Differentiating the equation to the ellipse (186), we have

$$\frac{dx}{dz} = -\frac{c^2 z}{a^2 x}.$$

And the subtangent being written $= s$, we have also,

$$s + x = \frac{c^2 z^2 + a^2 x^2}{a^2 x} = \frac{c^2}{x}.$$

In like manner from the equation to the circle (186), (distinguished by accenting the letters), we find,

$$\frac{dx_{,}}{dz_{,}} = -\frac{z_{,}}{x_{,}}, \quad \text{and } s_{,} + x_{,} = \frac{b^2}{x_{,}}.$$

But since the tangent is common to both

$$\frac{dx}{dz} = \frac{dx_{,}}{dz_{,}}, \quad \text{and } s + x = s_{,} + x_{,}.$$

Hence we have $x = x_{,} \dfrac{c^2}{b^2}$.

And in like manner we should find $z = z_{,} \dfrac{a^2}{b^2}$.

Substituting these values we can easily eliminate either $x_{,}$ or $z_{,}$ between the two equations (186) and thus obtain,

$$\frac{x_{,}}{b} = \cos \omega' x = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}},$$

$$\frac{z_{,}}{b} = \cos \omega' z = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

The following is a very brief outline of the deductions of Sir W. R. Hamilton, referred to before (190, 191), relative to the conoidal cusps and circles of contact.

If ρ be any radius, the centre being the pole, and ρ ρ' the particular values corresponding to the cusps whose inclinations are above determined; then, taking $x = \rho \cos \rho x$, $y = \rho \cos \rho y$, $z = \rho \cos \rho z$ on substituting in the wave-surface equation, it is at length reduced into the form referred to by Fresnel, viz.

$$\frac{1}{\rho^2} = \frac{1}{2}\left(\frac{1}{c^2} + \frac{1}{a^2}\right) + \frac{1}{2}\left(\frac{1}{c^2} - \frac{1}{a^2}\right) \cos (\rho \rho' \pm \rho \rho''),$$

which expresses the double value of the velocity for all directions of ρ except when it coincides with either ρ or ρ''; that is, when either $\rho \rho' = 0$, or $\rho \rho'' = 0$.

Then, if we write

$$r' = x \cos \rho' x + z \cos \rho z,$$

$$r'' = x \cos \rho'' x + z \cos \rho'' z,$$

so that r' r'' are the projections of ρ on ρ' and ρ'' the equation becomes,

$$1 = \frac{1}{2}\left(\frac{1}{c^2} + \frac{1}{a^2}\right)\rho^2 + \frac{1}{2}\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\left(r'r'' \pm \sqrt{\rho^2 - r'^2}\,\sqrt{\rho^2 - r''^2}\right).$$

Having the equation in this form, if we transform to new co-ordinates $x_{,}$ $y_{,}$ $z_{,}$ still in the same planes, but making the line ρ' the positive semi-axis of z. That is,

$$x = x_{,} \cos \rho' z + z_{,} \cos \rho' x,$$

$$y = y_{,},$$

$$z = z_{,} \cos \rho' z - x_{,} \cos \rho' x,$$

$$\rho^2 = x_{,}^2 + y_{,}^2 + z_{,}^2,$$

$$r' = z_{,},$$

$$r'' = x_{,} \sin \rho' \rho'' + z_{,} \cos \rho' \rho''.$$

The equation becomes,

$$1 = \begin{cases} \dfrac{z_{\prime}^2}{b^2} + \dfrac{1}{2} z_{\prime} \left(\dfrac{1}{c^2} - \dfrac{1}{a^2}\right) \sin \rho'\rho'' \\ \quad + \dfrac{1}{2} \quad \left(\dfrac{1}{c^2} + \dfrac{1}{a^2}\right) (x_{\prime}^2 + y_{\prime}^2) \\ \quad \pm \dfrac{1}{2} \quad \left(\dfrac{1}{c^2} - \dfrac{1}{a^2}\right) \sqrt{x_{\prime}^2 + y_{\prime}^2} \sqrt{(z_{\prime} \sin \rho'\rho'' - x_{\prime} \cos \rho'\rho'')^2 + y_{\prime}^2}. \end{cases}$$

Near the end of the radius ρ', that is, supposing $x_{\prime}$ and $y_{\prime}$ very small so that we may neglect terms of *two* dimensions, and having $z_{\prime} = b$, the equation is reduced to

$$z_{\prime} = b - \frac{1}{2} b^3 \sqrt{\frac{1}{c^2} - \frac{1}{b^2}} \sqrt{\frac{1}{b^2} - \frac{1}{a^2}} \left(x_{\prime} \pm \sqrt{x_{\prime}^2 + y_{\prime}^2}\right),$$

which is the equation to the *tangent cone.*

In the directions of single-normal-velocity the equation assumes a form *analogous* to that for single-ray-velocity above, viz.:

$$\omega^2 = \frac{1}{2}(a^2 + c^2) + \frac{1}{2}(a^2 - c^2) \cos(\omega\omega' \pm \omega\omega'').$$

Here, in like manner, if we transform to new axis $x_{\prime\prime}$ $y_{\prime\prime}$ $z_{\prime\prime}$ in the same planes, taking ω' as the positive semi-axis of $z_{\prime\prime}$, that is,

$$x = x_{\prime\prime} \cos \omega' z + z_{\prime\prime} \cos \omega' x,$$

$$y = y_{\prime\prime},$$

$$z = z_{\prime\prime} \cos \omega' z + x_{\prime\prime} \cos \omega' x.$$

The equation to the wave-surface becomes,

$$\left(x_{\prime\prime}^2 + y_{\prime\prime}^2 + z_{\prime\prime} \frac{x_{\prime\prime}}{b^2} \sqrt{a^2 - b^2} \sqrt{b^2 - c^2}\right)^2$$

$$= \left\{(a^2 + c^2)\rho^2 + (a^2 - c^2) r'r'' - a^2c^2\left(1 + \frac{z_{\prime\prime}^2}{b^2}\right)\right\}\left(1 - \frac{z_{\prime\prime}^2}{b^2}\right),$$

which, on making $z_{\prime\prime} = b$, gives

$$x_{\prime\prime}^2 + y_{\prime\prime}^2 + \frac{x_{\prime\prime}}{b} \sqrt{a^2 - b^2} \sqrt{b^2 - c^2} = 0.$$

The equation to a circle in the plane perpendicular to ω' at its extremity when its length is b, which is therefore the circle of contact; whose diameter also is,

$$\frac{\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}}{b}.$$

This also coincides with a formula of Fresnel somewhat differently expressed.

CALCULATION OF INDICES.

For the assistance of those who may wish to verify the calculations of indices, or to apply the method in other cases, it may be convenient to subjoin the actual computation of the constants A and B, common to all media, as well as to give one instance of the entire work of calculation for a particular medium.

For the calculation of the constants:

writing
$$h = l^2_H + 2^2_B, \qquad m = l^2_H - l^2_B,$$
$$n_{,} = l^2_H - 2\,l_{,}^2 + l^2_B, \qquad p_{,} = l_{,}^2 - l^2_B,$$
$$\log(-A_{,}) = \log n_{,} - \log m,$$
$$\log(-B_{,}) = \log p_{,} + \log A_{,} - \log m.$$

Then from table (269) we have

$$m = 31.170, \qquad \log m = 1.49373,$$
$$h = 62.146, \qquad \log h = 1.79341.$$

Whence for the several rays C, D, E, G, and also for the limit, and for the mean ray of heat (where $\lambda = .000079$) which we call K, the following values result:

Ray.	$n_{,}$	$\log n_{,}$	$p_{,}$	$\log p_{,}$
Limit	62.146	1.79341	− 15.488	1.18999
K	58.942	1.77042	− 13.886	1.14257
C	28.052	1.44796	1.559	0.19284
D	19.868	1.29815	5.651	0.75212
E	9.278	0.96745	10.946	1.03925
G	− 17.264	1.23714	24.217	1.38412

From these directly result the numbers given in (270) and (348), by means of the above formula.

$D = \mu_F - \mu_B$ $= .0195$		Indices assumed, μ_B 1.5320 μ_F 1.5515 μ_H 1.5749	$\mu_B + \mu_H = 3.1069$ $2\mu_F = 3.1030$ $D' = \mu_B + \mu_H - 2\mu_F$ $= .0039$
$\log D = \bar{2}.2900$			$\log D' = \bar{3}.5911$
	$\log A + \log D$	$\mu_{,} = \mu_F \pm (AD + BD')$	$\log B + \log D'$
Limit	0.2997 $\bar{2}.5897$ ...	$AD = .0389$ $-.0038 = BD'$ $-.0351$ 1.5164	$\bar{1}.996$ .... $\bar{3}.587$
C	$\bar{1}.95433$ $\bar{2}.2443$	.0175 + .0001 − .0176 1.5339	$\bar{1}.6525$ $\bar{4}.2436$
D	$\bar{1}.80441$ $\bar{2}.0944$	.0124 + .0004 − .0128 1.5387	$\bar{1}.0628$ $\bar{4}.6539$
E	$\bar{1}.49646$ $\bar{3}.7864$	.0061 + .0004 − .0065 1.5450	$\bar{1}.0319$ $\bar{4}.6230$
G	$\bar{1}.74027$ $\bar{2}.0302$	.0108 + .0016 + .0124 1.5639	$\bar{1}.6338$ $\bar{3}.2249$

THE END.